Wealth distribution and output fluctuations[*]

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Abstract: *We explore the link between wealth inequality and output fluctuations in a general two-sector neoclassical growth model with endogenous labor and heterogeneous agents. When agents have homogeneous CRRA preferences and individual wealth is Pareto distributed a sufficiently large rise in the Gini index typically leads to an increase in endogenous fluctuations of output. For general economies, we show that under plausible conditions on the fundamentals, wealth inequality is still a destabilizing factor.*

Keywords: *Wealth Inequality, Pareto distribution, Gini index, Elastic Labor Supply, Endogenous Equilibrium Business Cycles.*

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1 Introduction

Is there a relationship between income or wealth inequality and output fluctuations? Recent data concerning the Latin American and the OECD countries, as well as the East Asian "tigers", suggest a positive answer. In 1990, the Gini coefficient of the distribution of income was on average 59.5% for Brazil, Chile, Mexico and Venezuela while it was 34% for the OECD countries. During the same decade, the average standard deviation of the rate of output growth was 5.9% for the mentioned Latin American countries and only 2.7% for the OECD. East Asian countries behaved similarly: Hong Kong, Korea, Taiwan and Singapore had on average a (small) Gini coefficient of 35.5% and a volatility of 2.8%. Building on these data, Breen and García-Peñalosa [13] show the existence of a significant positive correlation between a country's output volatility and income inequality.

In the present paper we explore the impact of the wealth distribution on output fluctuations from a theoretical perspective. Two remarks guide our modelling choices. First, we take the view that the link between wealth inequality and business cycle fluctuations needs first to be examined in a model without distortions and in the limit with no external stochastic disturbances. Indeed, following Benhabib and Nishimura [8], it is possible to link the standard notion of macroeconomic fluctuation (based on stochastic oscillations) to endogenous instability of deterministic models, through the concept of cyclic sets. [1] Secondly, it is known that instability easily occurs in perfectly competitive multi-sector growth models. Whenever endogenous leisure is a normal good, the simplest of such models is the two-sector optimal growth model *à la* Uzawa. Finally, we introduce heterogeneous agents and characterize the level of wealth inequality by the distribution of shares that the individuals have in the firms and by the distribution of individual endowments.

The message of the present paper is that, within this standard framework, in most realistic situations wealth inequality is a destabilizing factor. When agents have homogeneous CRRA preferences and individual wealth follows a Pareto distribution, a sufficiently large rise in the Gini index typically leads to an increase in endogenous fluctuations of output. For general economies, we provide a general characterization and then focus on two

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[1] Benhabib and Nishimura [8] show that introducing small stochastic shocks into a deterministic model characterized by periodic cycles generates cyclic sets.

configurations: i) heterogeneous CRRA preferences, and ii) homogeneous preferences characterized by non-linear individual absolute risk tolerance indices. In both cases, we show that under the plausible assumptions of a large enough social elasticity of intertemporal substitution in consumption and a low enough social elasticity of labor, an increase of wealth inequality generates output fluctuations.

The sharp results obtained in this paper contrast with some of the previous related literature, in particular with Ghiglino and Venditti [21] who also investigate the link between inequality and instability in a neoclassical model. However, their assumption of inelastic labor supply has two major drawbacks: 1) the model does not produce any significant relationship between inequality and fluctuations when agents have standard CRRA preferences, and 2) the role of the wealth distribution depends exclusively on the curvature of the absolute risk tolerance index. In the present model we allow for endogenous labor supply, implying that also the aggregate steady state depends on the wealth distribution. Interestingly, the shape of preferences loses its predominant role in the link between inequality in wealth and fluctuations, allowing a much better characterization.

Beside Ghiglino and Venditti [21], the impact of income and wealth inequality on output fluctuations has generally been addressed within models with externalities, increasing returns or price rigidities. These various forms of market imperfections introduce many degrees of freedom, which allows dynamic instability of the long run equilibrium, either deterministic or stochastic, but prevent clear-cut results. Indeed, in some cases inequality is shown to be a stabilizing factor, as in Herrendorf *et al.* [25] or Ghiglino and Sorger [20], while in others the effect appears to be the opposite. A good example here is Aghion *et al.* [1] where inequality in the form of unequal access to investment opportunities across agents results in output and investment fluctuations.

The rest of the paper is organized as follows: The methodology and the main results are briefly summarized in Section 2 while the model is introduced in Section 3. Section 4 is concerned with the definition of the equilibrium and the existence of a steady state at the individual and aggregate levels. Section 5 discusses the existence of endogenous business cycle fluctuations. In Section 6, the occurrence of output fluctuations is related to wealth inequality in the particular case in which preferences are CRRA and wealth is Pareto distributed. This result is extended to general preferences and wealth distributions in Section 7. Section 8 concludes the paper, and all the proofs are in a final Appendix.

2 Summary of the paper

Starting from the decentralized model with many agents, our methodology consists in aggregating heterogeneous preferences into a planner's utility function, which depends on a set of welfare weights. We solve the planner's problem and analyze the stability properties of the optimal growth path as a function of the welfare weights. The equilibrium welfare weights are obtained when the associated allocations saturate the budget constraints of all the consumers. Then we show that the equilibrium welfare weights are continuous functions of the initial conditions. Consequently, the local dynamic properties of the steady state of the general equilibrium model with heterogeneous agents and those of the planners' problem with welfare weights fixed at their steady state value are identical.

Building on Bosi *et al.* [12], we provide conditions on the technologies and the social utility function for the existence of endogenous business cycle fluctuations either damped in the long run or persistent through period-2 cycles. As initially shown by Benhabib and Nishimura [7], we need a capital-intensive consumption good sector in order to allow some oscillations of the capital stock "to get through" the Rybczinsky theorem. But the properties of preferences also matter: first, fluctuations in consumption levels along the equilibrium path require a high enough social elasticity of intertemporal substitution in consumption. Second, a low enough social elasticity of labor supply is necessary to prevent the agents from smoothing the fluctuations in their wage and capital incomes associated with the fluctuations in the capital stock.[2] These conditions translate into a high enough social absolute risk tolerance with respect to consumption and a low enough social absolute risk tolerance with respect to labor.

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[2] When the labor supply is highly elastic, fluctuations in the wage rate and the rental rate of capital can be compensated for by major modifications in the labor supply. The fluctuations in income are thus smoothed and the cycles can be eliminated. Conversely, when the labor supply is not very elastic, fluctuations in the capital stock generate fluctuations in incomes and cycles become persistent.

In Section 6 we show that a positive correlation between the degree of wealth inequality and output fluctuations is obtained when agents have homogeneous CRRA preferences and individual wealth follows a Pareto distribution. The basic intuition for this result is the following: when wealth inequality increases, the steady state values of the social absolute risk tolerance indices with respect to consumption and labor are changed. Under a Pareto distribution, these modifications are such that the social elasticity of intertemporal substitution in consumption increases more sharply than the social elasticity of labor when the degree of inequality rises. This behavior implies that the curvature of the social utility function decreases with respect to consumption but increases with respect to labor, affecting the planner's attitude towards risk. As a result, endogenous business cycle fluctuations become possible.

In Section 7 we extend this conclusion to general wealth distributions and general preferences. According to the Pigou-Dalton principle, wealth inequality rises when a bilateral transfer between two agents increases the income of the agent who is initially richer than the other. In general economies, generically the social absolute risk tolerance indices with respect to consumption and labor are non-linear and then a modification in the degree of inequality affects both the individual and the aggregate steady states. In turn, this affects the local stability properties of the equilibrium through the variations of the social indices. We focus our analysis to two circumstances in which the non-linearity of the social absolute risk tolerance indices occurs.

First, we consider the case in which individual absolute risk tolerance indices are linear (as with CRRA preferences) but agents have heterogeneous preferences. We show that a positive relationship between inequality and business cycle fluctuations occurs under two empirically plausible sets of conditions: either i) if the richer agent, who receives a positive transfer, is characterized by a higher risk-tolerance with respect to consumption but a lower risk tolerance with respect to labor than the poorer agent, provided the individual elasticities of intertemporal substitution in consumption are large enough and the individual elasticities of labor are low enough; or ii) if the richer agent, who receives a positive transfer, is characterized by a lower risk tolerance with respect to consumption but a higher risk tolerance with respect to labor than the poorer agent, provided the individual elasticities of intertemporal substitution in consumption are low enough.

Second, we show that the same kind of results hold with homogenous preferences but with non-linear individual absolute risk tolerance indices. Indeed, a positive relationship between inequality and business cycle fluctuations occurs: either i) with convex individual absolute risk tolerance indices provided the individual elasticities of intertemporal substitution in consumption are large enough while the individual elasticities of labor are low enough; or ii) with concave individual absolute risk tolerance indices provided the individual elasticities of intertemporal substitution in consumption are low enough.

Finally, we discuss the plausibility of these results in view of the empirical findings.

3 The model

3.1 Consumers

There are n agents and the total population is constant over time. In each period consumers provide elastically an amount of labor l_i, $i = 1, ..., n$, with $l_i \leq \bar{l}$ and $\sum_{i=1}^{n} l_i = \ell$, $\bar{l} > 0$ being the agent's endowment of labor. Furthermore, they receive a fixed endowment of the consumption good ω_i with $\sum_{i=1}^{n} \omega_i = \bar{\omega} \geq 0$. At the initial period $t = 0$, each agent i is also endowed with a share θ_i of the initial stock of capital k_0 with $\sum_{i=1}^{n} \theta_i = 1$. In order to simplify the formulation, we will assume that the n agents are ordered according to their initial capital endowment, *i.e.*, $\theta_i \leq \theta_j$ for $i < j$. Let $(\theta_i)_{i=1}^{n} = \theta$ be the vector of initial shares and $(\omega_i)_{i=1}^{n} = \omega$ be the vector of individual endowments. Consumers' preferences are characterized by a discounted additively separable utility function of the form

$$\mathcal{U}^i(x^i, \mathcal{L}^i) = \sum_{t=0}^{\infty} \delta^t \left[u_i(x_{it}) + v_i(\mathcal{L}_{it}) \right] \tag{1}$$

where $\delta \in (0, 1)$ is the discount factor, x_{it} the consumption of agent i at time t, $\mathcal{L}_{it} = \bar{l} - l_{it}$ its leisure at time t, and x^i, $\mathcal{L}^i$ are respectively its intertemporal streams of consumption and leisure. Agents may therefore differ with respect to their preferences and their initial wealth. Each instantaneous utility function satisfies the following basic restrictions:

Assumption 1. *$u_i(x_i)$ and $v_i(\mathcal{L}_i)$ are C^r with $r \geq 3$, such that $u_i'(x_i) > 0$, $v_i'(\mathcal{L}_i) > 0$, $u_i''(x_i) < 0$, $v_i''(\mathcal{L}_i) < 0$ for any $x_i > 0$, $\mathcal{L}_i > 0$, and satisfy the Inada conditions $\lim_{x_i \to 0} u_i'(x_i) = +\infty$, $\lim_{\mathcal{L}_i \to 0} v_i'(\mathcal{L}_i) = +\infty$.*

Denote by w_t the wage rate, r_t the gross rental rate of capital and p_t the price of investment good at time t, all in terms of the price of the consumption good. In a decentralized economy, an agent i maximizes his intertemporal utility function (1) subject to the intertemporal budget constraint

$$\sum_{t=0}^{\infty} R_t x_{it} = \sum_{t=0}^{\infty} R_t(\omega_i + w_t l_{it}) + \theta_i r_0 k_0 \quad \text{with } i = 1, ..., n, \tag{2}$$

where the discount factors R_t are defined as:

$$R_t = \prod_{\tau=0}^{t} \frac{1}{1 + d_\tau}$$

with d_t the common interest rate, which satisfies $d_0 = [r_0 - p_{-1}]/p_{-1}$ and $d_t = [r_t + (1-\mu)p_t - p_{t-1}]/p_{t-1}$ for any $t \geq 1$.[3]

3.2 Producers

We consider a two-sector economy with a consumption good y_0 and a capital good y. The consumption good is entirely consumed and the capital good partially depreciates in each period at a constant rate $\mu \in [0, 1]$. There are two inputs, capital and labor. Each good is produced with a standard constant returns to scale technology:

$$y_0 = f^0(k^0, l^0), \quad y = f^1(k^1, l^1)$$

with $k^0 + k^1 \leq k$, k being the total stock of capital, and $l^0 + l^1 \leq \ell$, ℓ being the total amount of labor.

Assumption 2. *Each production function $f^j : \mathbb{R}_+^2 \to \mathbb{R}_+$, $j = 0, 1$, is C^2, increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $f_1^j(0, x) = f_2^j(x, 0) = +\infty$, $f_1^j(+\infty, x) = f_2^j(x, +\infty) = 0$.*

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[3] This equation reflects the absence of arbitrage opportunities in a perfect foresight equilibrium. It is also called the *portfolio equilibrium condition* (see Becker and Boyd [3]). The difference between the equation evaluated at time $t = 0$ and $t \geq 1$ comes from the fact that at the initial date there is no residual capital coming from the previous period and in some sense we have $k_0 = y_{-1}$.

Note that by definition, as $l_1 \leq \ell \leq n\bar{l}$, we have $y \leq f^1(k,\ell) \leq f^1(k,n\bar{l})$. The monotonicity properties and the Inada conditions in Assumption 2 then imply that there exists $\bar{k} > 0$ such that $f^1(k,n\bar{l}) > k$ when $k < \bar{k}$ while $f^1(k,n\bar{l}) < k$ when $k > \bar{k}$. The set of admissible 3-uples (k,y,ℓ) is thus defined as follows

$$\mathcal{D} = \left\{(k,y,\ell) \in \mathbb{R}^3_+ | 0 \leq l \leq n\bar{l}, 0 \leq k \leq \bar{k},\ 0 \leq y \leq f^1(k,\ell)\right\} \tag{3}$$

It is easy to show that $\mathcal{D}$ is a compact, convex set.

There are two representative firms, one for each sector. For any given (k,y,ℓ), profit maximization in each representative firm is equivalent to solving the following problem of optimal allocation of productive factors between the two sectors:

$$\begin{array}{rcl} T(k,y,\ell) = & \max\limits_{(k^0,k^1,l^0,l^1)\in\mathbb{R}^4_+} & f^0(k^0,l^0) \\ & s.t. & y \leq f^1(k^1,l^1),\ k^0+k^1 \leq k,\ l^0+l^1 \leq \ell \end{array} \tag{4}$$

The social production function $T(k,y,\ell)$ describes the frontier of the production possibility set associated with interior temporary equilibria such that $(k,y,\ell) \in \mathcal{D}$, and gives the maximal output of the consumption good. It also summarizes the trade-off between production of the final good and productive investment. Under Assumption 2, for any $(k,y,\ell) \in \mathcal{D}$, $T(k,y,\ell)$ is homogeneous of degree one, concave and twice continuously differentiable.[4]

We formulate the aggregate profit maximization as follows

$$\max_{(k,y,\ell)\in\mathcal{D}} T(k,y,\ell) + py - rk - w\ell \tag{5}$$

For any $(k,y,\ell) \in int\mathcal{D}$, with $int\ \mathcal{D}$ denoting the interior of the set $\mathcal{D}$, the first order derivatives of the social production function give

$$T_1(k,y,\ell) = r,\ \ T_2(k,y,\ell) = -p,\ \ T_3(k,y,\ell) = w \tag{6}$$

4 Competitive equilibrium and Pareto optimum

From the first welfare theorem, we know that every competitive equilibrium obtained in the decentralized economy is a Pareto optimal allocation. Let

$$\Delta = \left\{\eta_1,\ldots,\eta_n | \eta_i \geq 0 \text{ and } \sum_{i=1}^n \eta_i = 1\right\}$$

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[4] See Benhabib and Nishimura [6].

be the unit simplex of $\mathbb{R}^n$. A Pareto optimal allocation is a solution to the following planner's problem for a given vector of nonnegative welfare weights $\eta = (\eta_1, \ldots, \eta_n) \in \Delta$:

$$\begin{aligned} \max_{\{x_{it}, l_{it}, y_t\}_{t \geq 0}} \quad & \sum_{i=1}^{n} \eta_i \sum_{t=0}^{\infty} \delta^t \left[u_i(x_{it}) + v_i(\bar{l} - l_{it}) \right] \\ s.t. \quad & \sum_{i=1}^{n} x_{it} = \bar{\omega} + T(k_t, y_t, \ell_t) \\ & \sum_{i=1}^{n} l_{it} = \ell_t \\ & k_{t+1} = y_t + (1-\mu) k_t \text{ with } k_0 \text{ given,} \end{aligned} \tag{7}$$

The solution to this program depends on the vector η and on k_0. The set of Pareto optima is obtained when η spans Δ. As markets are complete and under Assumptions 1 and 2, the second theorem of welfare economics also holds: any Pareto efficient allocation can be decentralized as a competitive equilibrium with transfer payments. We can then characterize any equilibrium with transfers by solving the weighted dynamic optimization program (7).[5] A given competitive equilibrium is then obtained for a η such that the associated allocations saturate the budget constraint of all the consumers.

In order to simplify the analysis, we formulate the weighted dynamic optimization program (7) in reduced form. Let $U(x, \ell)$ be a social utility function such that for $\eta = (\eta_1, \ldots, \eta_n) \in \Delta$

$$\begin{aligned} U(x_t, \ell_t) \quad = \quad & \max_{\{x_{it}, l_{it}\}_{t \geq 0}} \quad \sum_{i=1}^{n} \eta_i \left[u_i(x_{it}) + v_i(\bar{l} - l_{it}) \right] \\ & \qquad s.t \quad \sum_{i=1}^{n} x_{it} = x_t \text{ and } \sum_{i=1}^{n} l_{it} = \ell_t \end{aligned} \tag{8}$$

The value function $U(x, \ell)$ can be characterized as follows:[6]

Lemma 1. *Under Assumption 1, the value function of program (8) is additively separable,* i.e. *$U(x, \ell) = u(x) - v(\ell)$ with $u(x)$ and $v(\ell)$ some C^r functions with $r \geq 3$, such that $u'(x) > 0$, $v'(\ell) > 0$, $u''(x) < 0$, $v''(\ell) > 0$ for any $x > 0$, $\ell > 0$, and* $\lim_{x \to 0} u'(x) = +\infty$, $\lim_{\ell \to n\bar{l}} v'(\ell) = +\infty$.

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[5] This approach was pioneered by Negishi [30] and applied to dynamic models by Bewley [9] and Kehoe *et al.* [26] among others.

[6] The proof is straightforward and available upon request.

We then define the indirect social utility function

$$V(k_t, k_{t+1}, \ell_t) = u(\bar{\omega} + T(k_t, k_{t+1} - (1-\mu)k_t, \ell_t)) - v(\ell_t) \tag{9}$$

Considering from the capital accumulation equation in (7) that $k_{t+1} \geq (1-\mu)k_t$, the set of admissible paths (k_t, k_{t+1}, ℓ_t) is derived from (3) as follows

$$\begin{aligned}\tilde{\mathcal{D}} = \Big\{ & (k_t, k_{t+1}, \ell_t) \in \mathbb{R}^3_+ | 0 \leq \ell_t \leq n\bar{l}, \\ & 0 \leq k_t \leq \tilde{k},\ (1-\mu)k_t \leq k_{t+1} \leq f^1(k_t, \ell_t) + (1-\mu)k_t \Big\}\end{aligned}$$

with $\tilde{k} > 0$ such that $f^1(\tilde{k}, n\bar{\ell}) = \mu\tilde{k}$, and the planner's problem is equivalent to

$$\begin{array}{cl} \max\limits_{\{k_t, \ell_t\}_{t\geq 0}} & \sum\limits_{t=0}^{+\infty} \delta^t V(k_t, k_{t+1}, \ell_t) \\ s.t. & (k_t, k_{t+1}, \ell_t) \in \tilde{\mathcal{D}} \\ & k_0 \text{ given} \end{array} \tag{10}$$

Note that the solution depends on k_0.

In the present framework it is a standard result that the set of interior Pareto optima is the set of $\{k_t, \ell_t\}_{t\geq 0}$ that are solutions to the following system of Euler equations

$$V_2(k_t, k_{t+1}, \ell_t) + \delta V_1(k_{t+1}, k_{t+2}, \ell_{t+1}) = 0 \tag{11}$$

$$V_3(k_t, k_{t+1}, \ell_t) = 0 \tag{12}$$

and that satisfy the transversality condition

$$\lim_{t\rightarrow+\infty} \delta^t k_t V_1(k_t, k_{t+1}, \ell_t) = 0$$

Note that using (6) and (9), the Euler equations become:

$$-u'(x_t)p_t + \delta u'(x_{t+1})\left[r_{t+1} - (1-\mu)p_{t+1}\right] = 0 \tag{13}$$

$$u'(x_t)w_t - v'(\ell_t) = 0 \tag{14}$$

Our methodology consists in providing a local stability analysis of the optimal path in a neighborhood of the steady state, which is obtained as a stationary solution of the Euler equations. Within an optimal growth model with heterogeneous agents, the steady state has to be considered along two dimensions. At the aggregate level, an interior steady state is a sequence $(k_t, y_t, \ell_t) = (k^*, \mu k^*, \ell^*)$, $\forall t \geq 0$, with $x_t = x^* = \bar{\omega} + T(k^*, \mu k^*, \ell^*)$, $p_t = p^* = -T_2(k^*, \mu k^*, \ell^*)$, $r_t = r^* = T_1(k^*, \mu k^*, \ell^*)$ and $w_t = w^* = T_3(k^*, \mu k^*, \ell^*)$, that solves the Euler equations (13)-(14). Since $T(k, y, \ell)$ is a linear homogeneous function, and denoting $\kappa = k/\ell$, an aggregate steady state can also be defined as a pair (κ^*, ℓ^*).

At the individual level, an interior steady state for agent i is a sequence of consumption and labor supply $(x_{it}, l_{it}) = (x_i^*, l_i^*)$ that solves the first order conditions corresponding to the individual maximization of the intertemporal utility function (1) subject to the intertemporal budget constraint (2). Of course, the set of individual steady states (x_i^*, l_i^*), $i = 1, \ldots, n$, satisfy $x^* = \sum_{i=1}^n x_i^*$ and $\ell^* = \sum_{i=1}^n l_i^*$. Moreover, the stationary values at the individual and aggregate levels depend on the initial distribution of capital $\theta = (\theta_i)_{i=1}^n$ and of individual endowments $\omega = (\omega_i)_{i=1}^n$.

Theorem 1. *Let Assumptions 1 and 2 hold. Denote* $\kappa = k/\ell$ *and* $\vartheta = [1 - \delta(1-\mu)]^{-1}$.

i) There exists a unique aggregate steady state (κ^*, ℓ^*)*, called the Modified Golden Rule.*

ii) If $v_i'(\bar{l}) < u_i'(\omega_i + (1-\delta)\vartheta r^* \kappa^* \ell^* \theta_i)$ *for any* $i = 1, \ldots, n$*, there exist unique interior steady state values for the individual consumptions* $x_i^*(\theta, \omega) \in (0, x^*)$ *and labor supplies* $l_i^*(\theta, \omega) \in (0, \ell^*)$*, which are* C^1*-functions of* θ *and* ω.[7] *Moreover,* ℓ^* *and* x^* *are* C^1*-functions of* θ *and* ω.

Remark 1. Using the linear homogeneity of $T(k, y, \ell)$, we obtain the expression for the wage rate at the steady state $w^* = x^*/\ell^* - \bar{\omega}/\ell^* - (1-\delta)\vartheta r^* \kappa^*$. It follows that x_i^* and l_i^* can also be expressed as functions of the aggregate steady state values for consumption x^* and labor ℓ^*.

5 Endogenous competitive business cycles

Near the steady state the behavior of the non-linear dynamic system (13)-(14) is equivalent to the behavior of the linearized system. The local stability properties of the steady state are then related to the eigenvalues of the Jacobian matrix associated with the linearized system. On the one hand, the characteristic roots depend on the first and second order derivatives of

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[7] In a similar but aggregate model, Sorger [36] shows that a continuum of individual stationary equilibria occurs. In our framework, as the steady state is obtained for a given set of welfare weights η, the same result is obtained when η spans Δ. In other words, there exists a different individual stationary equilibrium for each $\eta \in \Delta$.

the social production function $T(k, y, \ell)$ through the difference in relative capital intensity across sectors

$$b(k, y, \ell) = \frac{l^0}{T}\left(\frac{k^1}{l^1} - \frac{k^0}{l^0}\right) \tag{15}$$

Note that $b(k, y, \ell) > (<)\ 0$ if and only if the investment (consumption) good is capital-intensive. As shown in Benhabib and Nishimura [7] and Bosi *et al.* [12], the existence of endogenous fluctuations requires $b(k, y, \ell) < 0$.

On the other hand, the characteristic roots depend on the first and second order derivatives of the social utility function through standard curvature indices.

Definition 1. *Let $U(x, \ell) = u(x) - v(\ell)$ be the social utility function, as defined by (8) and Lemma 1, and $\rho(x) = -u'(x)/u''(x) > 0$, $\gamma(\ell) = v'(\ell)/v''(\ell) > 0$ be the social absolute risk tolerance respectively for consumption and labor.*

As shown in Wilson [40], the social absolute risk tolerance indices are obtained from the individual ones as follows:

$$\rho(x) = \sum_{i=1}^{n} \rho_i(x_i(\theta)), \quad \gamma(\ell) = \sum_{i=1}^{n} \gamma_i(l_i(\theta)) \tag{16}$$

with

$$\rho_i(x_i) = -u_i'(x_i)/u_i''(x_i), \quad \gamma_i(l_i) = -v_i'(\bar{l} - l_i)/v_i''(\bar{l} - l_i) \tag{17}$$

For given discount factor δ and capital intensity difference b, the local stability properties of the steady state also depend on ℓ^*, $\rho(\ell^* T^*)$ and $\gamma(\ell^*)$.[8]

As explained in Section 4, our strategy consists in characterizing the competitive equilibrium through the analysis of the Pareto optimal solution of the planner's program (10). The equilibrium path is then the solution to the planner's intertemporal maximization problem, where the planner's utility, or social utility, is the sum of the individual utilities weighted by the welfare weights. Consequently, the social utility function depends on the welfare weights, which themselves depend on the equilibrium allocations that in turn depend on the distributions of initial capital shares and individual endowments. This means that without regularity properties of the welfare weights, the local stability of the steady state cannot be obtained directly from the local stability of the planner's optimum with fixed welfare weights.

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[8] Note that the model with inelastic labor is obtained by assuming $\gamma(\ell^*) = 0$.

We have shown in Theorem 1 that the aggregate and individual steady states are continuous functions of $\theta = (\theta_i)_{i=1}^n$ and $\omega = (\omega_i)_{i=1}^n$. As shown in Kehoe *et al.* [26] and Santos [35], if the value function of the dynamic optimization program (10) is twice continuously differentiable, the welfare weights are continuous functions of θ and ω, and the local dynamic properties of the competitive equilibrium can be analyzed from the planner's problem defined in terms of the social utility function with welfare weights fixed at their steady state value. Indeed, local stability means that with initial conditions slightly away from the steady state, the welfare weights will be close to their steady state values. The next Proposition gives a sufficient condition for the continuity of the welfare weights.

Proposition 1. *Under Assumptions 1 and 2, the welfare weights* $(\eta_1, \dots, \eta_n) \in \Delta$ *are continuous functions of the initial individual shares of capital* $\theta = (\theta_i)_{i=1}$ *and of individual endowments* $\omega = (\omega_i)_{i=1}^n$ *if for any* $(k_t, k_{t+1}, \ell_t) \in int\mathcal{D}$*:*

$$\begin{aligned} & T_2(k_t, k_{t+1} - (1-\mu)k_t, \ell_t) \\ + \quad & b(k_t, k_{t+1} - (1-\mu)k_t, \ell_t) T_1(k_t, k_{t+1} - (1-\mu)k_t, \ell_t) \neq 0 \end{aligned} \tag{18}$$

A sufficient condition for (18) to hold is $b(k_t, k_{t+1} - (1-\mu)k_t, \ell_t) \leq 0$, i.e., *the consumption good is capital-intensive.*

From (16) and Proposition 1 we derive that $\rho(x)$ and $\gamma(\ell)$ are continuous functions of the initial individual shares of capital $\theta = (\theta_i)_{i=1}^n$ and of individual endowments $\omega = (\omega_i)_{i=1}^n$. As a direct consequence, we finally conclude that the dynamic properties of the competitive equilibrium can be analyzed from the planner's problem defined in terms of the social utility function with fixed welfare weights.

Proposition 2. *Under Assumptions 1 and 2, let condition (18) hold. Then the stability properties of the steady state of the general equilibrium model with equilibrium welfare weights and of the optimal growth model with appropriate fixed welfare weights are equivalent.*

Building on Proposition 2, we consider from now on capital intensity configurations under which condition (18) holds and we pursue our analysis of the equilibrium path of the optimal growth model with welfare weights fixed at their steady state values. Our objective is to derive a relationship between, on the one hand, the local dynamic properties of the equilibrium path in a neighborhood of the steady state and, on the other hand, the degree of inequality in the economy based on the distribution of individual wealth, which depends on the distribution of capital and individual endowments across agents.

We are now ready to relate the stability of the steady state with the properties of the fundamentals. First, we focus on the role of technology. As shown in Bosi *et al.* [12], the existence of endogenous fluctuations in a two-sector optimal growth model with elastic labor requires a capital-intensive consumption good with $b \in (-1/(1-\mu), 0)$. The intuition for this fact, initially provided by Benhabib and Nishimura (1985), is based on the Rybczinsky theorem: an instantaneous increase in the capital stock k_t implies a decrease in the output of the capital good y_t, which lowers the investment and the capital stock in the next period k_{t+1}. But, from the same mechanism, this decrease in k_{t+1} implies an increase in the output of the capital good y_{t+1}, which increases the investment and the capital stock in period $t+2$, k_{t+2}. Fluctuations in the capital stocks are obtained if this mechanism is strong enough with respect to the depreciation rate of capital.[9]

Second, we focus on the role of preferences through the absolute risk tolerance indices. Indeed, the existence of persistent fluctuations is conditional on two main properties. On the one hand, the agents have to accept fluctuations in their consumption levels and thus need a great enough social elasticity of intertemporal substitution in consumption, i.e., a high enough social absolute risk tolerance with respect to consumption. On the other hand, as labor is supplied endogenously, a low enough social elasticity of the labor supply, i.e., a low enough social absolute risk tolerance with respect to labor, is necessary. Indeed this prevents the agents from smoothing the fluctuations in their wage and capital incomes implied by the fluctuations in the capital stock.

The following two Propositions, which rest on Proposition 4 in Bosi *et al.* [12], will be used to obtain our main results stated in the next Section. The first one deals with the existence of damped fluctuations.

Proposition 3. *Under Assumptions 1 and 2, let the consumption good be*

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[9] From the capital accumulation equation in (7), we get $dk_{t+1}/dk_t = b^{-1} + 1 - \mu$.

capital-intensive with $b \in (-1/(1-\mu), -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0)$*. Then there exist* $\rho_c > 0$ *and* $\gamma_c > 0$ *such that:*

1. If $\rho(x^*) \leq \rho_c$*, for any* $\gamma(\ell^*)$*, the steady state is saddle-point stable with monotone convergence.*

2. If $\rho(x^*) > \rho_c$*, the steady state is saddle-point stable with monotone convergence when* $\gamma(\ell^*) > \gamma_c$ *while it is is saddle-point stable with oscillating convergence when* $\gamma(\ell^*) < \gamma_c$*.*

Note that if ρ_c is too large or γ_c is too small, the steady state is saddle-point stable with monotone convergence for any $\rho(x^*)$ and $\gamma(\ell^*)$.

The second Proposition focuses on the existence of persistent oscillations.

Proposition 4. *Under Assumptions 1 and 2, let the consumption good be capital-intensive with* $b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu)))$*. Then there exist* $\rho_f > 0$ *and* $\gamma_f > 0$ *such that:*

1. If $\rho(x^*) \leq \rho_c$*, for any* $\gamma(\ell^*)$*, the steady state is saddle-point stable with monotone convergence.*

2. If $\rho(x^*) \in (\rho_c, \rho_f]$*, then the steady state is saddle-point stable with monotone convergence when* $\gamma(\ell^*) > \gamma_c$ *while it is is saddle-point stable with oscillating convergence when* $\gamma(\ell^*) < \gamma_c$*.*

3. If $\rho(x^*) > \rho_f$*, the steady state is saddle-point stable with monotone convergence when* $\gamma(\ell^*) > \gamma_c$*, saddle-point stable with oscillating convergence when* $\gamma(\ell^*) \in (\gamma_f, \gamma_c)$ *and becomes locally unstable with oscillating divergence when* $\gamma(\ell^*) < \gamma_f$*. Moreover,* γ_f *is a flip bifurcation value and there generically exist period-two cycles, in a left (or right) neighborhood of* γ_f*, which are saddle-point stable (or unstable).*

Note that if ρ_f is too large or γ_f is too small, the steady state is saddle-point stable with monotone or oscillating convergence for any $\rho(x^*)$ and $\gamma(\ell^*)$.

To conclude, Propositions 3 and 4 provide a complete characterization of the local dynamics when $b \in (-1/(1-\mu, 0)$ for all values of the fundamental parameters. For values of $b \notin (-1/(1-\mu, 0)$, the steady state is saddle-point stable with monotone convergence for any $\rho(x^*)$ and $\gamma(\ell^*)$.

6 The leading case: Pareto distributed wealth and CRRA preferences

In this section we analyze a simple and arguably plausible specification. We assume that agents have homogeneous CRRA preferences and that individual wealth follows a Pareto distribution. On the one hand, the assumption on preferences is common in macroeconomics and most empirical findings show that it is a decent approximation of the real world. On the other hand, most OECD's economies, including the US, have been characterized over the last century by skewed distributions of income and wealth with relatively large top shares[10] and with heavy upper tails.[11]. The Pareto distribution has these features even though for low individual wealth values, the log-normal distribution provides a better fit.

In this section we restrict the analysis to the family of Pareto wealth distributions and investigate the effect of increasing inequalities on the dynamics within this family.[12] More precisely, we fix the population size and the technologies while we adjust the capital shares $(\theta_i)_{i=1}^n$ and the individual endowments $(\omega_i)_{i=1}^n$ such that at equilibrium the individual wealth follows a Pareto distribution.

As the present model is specified for a discrete and finite set of individuals, we need to focus on the discrete version of the Pareto distribution, which is the Zeta distribution. For a distribution with discrete and countable support $S = \{x_1, x_2, ...\}$, the associated "probability measure" μ is completely determined by $\mu(x_1), \mu(x_2), ...$ In the case of the Zeta distribution S is normalized in a way such that $S = \{1, 2, 3, ...\}$.

Definition 2. *The Zeta distribution* $\Gamma(\alpha)$ *of parameter* $\alpha > 1$ *is characterized by the "probability measure"* μ *defined for natural values* $s \in \mathbb{N}$ *by* $\mu(s) = \frac{s^{-\alpha}}{\sum_{j=1}^{\infty} j^{-\alpha}}$.

The fact that the underlying economy involves a countable and finite population implies that in our model $\mu(s)n$ can only take integer values,

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[10] See Piketty and Saez [32], Wolff [41, 42].

[11] See Nirei and Souma [31].

[12] Another weakness is that Pareto distributions have unbounded support, which generates a bias when compared to the data.

which is generically not true when μ is given by Definition 2. However, for a sufficiently large population n, the Zeta distribution can be approximated with an arbitrary precision by a distribution $\Gamma_n(\alpha)$ of measure μ_n (of finite support) such that $\mu_n(s)n$ takes integer values for all s and $\Gamma_n(\alpha)$ is in a neighborhood of $\Gamma(\alpha)$.

Definition 3 *Let* $\{\Gamma_n(\alpha)\}_{n>0}$ *be a sequence of distributions with probability measure* μ_n *such that for any* $s \in S$*, when* $\mu(s)n \notin \mathbb{N}$*,* $\mu_n(s)$ *is defined by* $|\mu_n(s) - \mu(s)| \leq 1/2n$ *with* $\mu_n(s)n \in \mathbb{N}$*, and when* $\mu(s)n \in \mathbb{N}$*,* $\mu_n(s) = \mu(s)$*.*

Along the steady state the support S is obtained by letting $s = \frac{\Omega}{\Omega_{\min}}$ where Ω is the individual wealth and $\Omega_{\min}$ is the minimum of Ω, *i.e.*, the subsistence level of wealth. This implies that agents are grouped in classes with the same wealth and that individual wealth only takes values which are multiples of $\Omega_{\min}$.

In the case of Pareto distributed wealth, the Gini index G is related to the exponent α of the Pareto distribution by $G = 1/(2\alpha - 1)$. Therefore, within this class of wealth distributions, high α values correspond to more equal societies. Let $G(\Gamma_n(\alpha))$ be the Gini index of the economy $\Gamma_n(\alpha)$ previously defined.

Lemma 2. *For any* $\alpha > 1$*, there exists a sequence of economies with normalized wealth distributions* $\{\Gamma_n(\alpha)\}_{n>0}$*. Furthermore, each* $\Gamma_n(\alpha)$ *is implementable with an appropriate choice of* $(\theta_i)_{i=1}^n$ *and* $(\omega_i)_{i=1}^n$*.*

The previous Lemma characterizes the family of economies considered in this section as well as their wealth distribution. This family consists of economies that are the closest approximation to the Pareto distribution in the present environment.

We now introduce two alternative assumptions concerning the capital intensity difference.

Assumption 3 . *The consumption good is capital-intensive with* $b \in (-1/(1-\mu), -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0)$*.*

Assumption 4. *The consumption good is capital intensive with* $b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu)))$. *Moreover,* γ_f *is a flip bifurcation value giving rise to saddle-point stable period-two cycles in its left neighborhood.*

These assumptions are motivated by Propositions 3 and 4. Assumption 3 is linked to the existence of damped fluctuations while Assumption 4 concerns the occurrence of persistent fluctuations, *i.e.*, saddle-point stable period-two cycles. Although the second part of Assumption 4 concerns non-trivial restrictions on the non-linear part of the Euler equation, a number of robust examples of saddle-point stable period-two cycles have been provided by Boldrin and Deneckere [11] and Mitra and Nishimura [28].

Finally, we assume that agents are homogeneous with respect to their utility function which has the standard CRRA formulation

$$u_i(x_i) = \frac{x_i^{1-\sigma}}{1-\sigma} \text{ and } v_i(\mathcal{L}_i) = \frac{(\bar{l}-\mathcal{L}_i)^{1+\gamma}}{1+\gamma} = \frac{l_i^{1+\gamma}}{1+\gamma}$$

with $\sigma, \gamma, > 0$ and $\mathcal{L}_i = \bar{l} - l_i$. Note that this formulation implies that

$$\rho_i(x_i) = \frac{x_i}{\sigma} \text{ and } \gamma_i(l_i) = \frac{l_i}{\gamma}$$

Let $E(\Gamma_n(\alpha))$ be the economy with wealth distribution $\Gamma_n(\alpha)$. We are now ready to state our main result:

Theorem 2. *Let Assumption 2 hold and agents be homogeneous with CRRA preferences. Then for any interior equilibrium and* n *sufficiently large, there exists* $\bar{\alpha} \in [1, +\infty) \cup \{+\infty\}$ *and a level of wealth inequality characterized by a Gini index* $\bar{G} = 1/(2\bar{\alpha} - 1) \in [0, 1]$ *such that one of the following cases holds:*

1 - If Assumption 3 is satisfied, the steady state is saddle-point stable with monotone convergence for any economy $E(\Gamma_n(\alpha))$ *such that* $G(\Gamma_n(\alpha)) < \bar{G}$ *and is saddle-point stable with oscillations otherwise.*

2 - If Assumption 4 is satisfied, the steady state is saddle-point stable with oscillating convergence for any economy $E(\Gamma_n(\alpha))$ *such that* $G(\Gamma_n(\alpha)) < \bar{G}$ *and is unstable otherwise. Moreover, there generically exist saddle-point stable period-two cycles, in a right neighborhood of* $\bar{G}$.

3 - For given technologies, there exists an open set Ξ *of values of* (σ, γ) *such that inequality do matter, i.e.,* $\bar{G} \in (0, 1)$.

Remark 2 *In the cases not covered by Assumptions 3 and 4 or if* $\bar{G} = 0$ *or* $\bar{G} = 1$*, wealth inequality has no impact on the local dynamic properties.*

Theorem 2 states that when the wealth distribution is Pareto and the agents are homogeneous with CRRA preferences, a rise in wealth inequality can generate endogenous fluctuations but can never stabilize the economy by ruling out business cycles.

The intuition of this result is as follows. Since agents have homogeneous preferences, the open set Ξ is obtained when the individual elasticity of intertemporal substitution $1/\sigma$ is large enough to guarantee a social absolute risk tolerance index such that $\rho(x^*) > \rho_c$ or ρ_f. This allows the agents to substitute consumption over time in order to smooth their level of utility when fluctuations arise. However, our result does not depend on the fact that a rise in wealth inequality generates intertemporal reallocations of consumption, but rather on the fact that a larger wealth inequality implies intertemporal reallocations of labor, which always decrease the aggregate elasticity of labor at the equilibrium. As shown by Propositions 3 and 4, this mechanism generates endogenous fluctuations.

Remark 3 *When labor is inelastic, as in Ghiglino and Venditti [21], the elasticity of labor is equal to zero and inequality does not matter. Indeed, the only possibility for the above mechanism to hold is to go through variations of the social risk tolerance index* $\rho(x^*)$ *when the degree of inequality increases. But in that case aggregate consumption* x^* *does not depend on the distribution of wealth, and both* $\rho(x^*)$, ρ_c, ρ_f *are linear in* x^*.

7 The impact of wealth inequality on output fluctuations in general economies

In the previous section we analyzed the simple case of homogeneous agents with CRRA preferences and Pareto distributed wealth. In this section we explore the link between wealth inequality and stability in a more general setting. However, for general wealth distributions, the Gini index does not provide the needed link between the absolute risk tolerance indices and inequality. We instead adopt a simple definition of increasing inequality based on the Pigou-Dalton transfer principle.

On the basis of Propositions 3 and 4 we need a measure of wealth inequality that can be linked to the steady state values of the absolute risk tolerance indices $\rho(x^*)$ and $\gamma(\ell^*)$. Clearly, we know that the aggregate consumption x^* and labor ℓ^* are functions of the distributions $\theta = (\theta_i)_{i=1}^n$ and $\omega = (\omega_i)_{i=1}^n$. Without lack of generality we consider only variations in capital shares θ and let $\omega_i = \omega_j$ for all i, j. We focus on the effects of bilateral transfers between two agents, say i, j, in which agent i initially owns a larger share of capital than agent j. According to the Pigou-Dalton transfer principle, inequality increases if agent i ends up holding an even larger share while agent j, who initially already owns a lower share of capital, ends up with an even lower share.

Definition 4. *Assume that the n agents are ordered according to their initial share of capital in decreasing order,* i.e., *$\theta_i \geq \theta_j$ for $i < j$. Consider a benchmark economy A characterized by a distribution $\theta^A = (\theta_k^A)_{k=1}^n$. There is a larger inequality in some economy B if the associated distribution $\theta^B = (\theta_k^B)_{k=1}^n$ is such that $\theta_i^A < \theta_i^B$ and $\theta_j^A > \theta_j^B$ for some agents i, j such that $j > i$, while for the remaining agents $k \neq i, j$, θ_k is unaffected. Such a case in which economy B is more unequal than economy A is denoted $\theta^A \prec_I \theta^B$.*

In Ghiglino and Venditti [21], inequality is defined as in Rotschild and Stiglitz [34]. Indeed, when labor is inelastic, the aggregate steady state is invariant with respect to the distribution of wealth across agents. A spread in the distribution of consumers' wealth decreases or increases the value of $\rho(x^*)$ depending on whether $\rho(x)$ is concave or convex. Assuming a convex absolute risk tolerance index is the only way to increase the value of $\rho(x^*)$ with a mean-preserving spread of capital shares. On the contrary, here the labor supply is endogenous and the aggregate steady state depends on the distribution of initial shares of capital. We are forced to consider a simpler notion of inequality based on bilateral transfers. Clearly, when the mean is preserved, these two notions are compatible.

In order to study the consequences of bilateral transfers on the aggregate steady state, we introduce two elasticities characterizing the agents' preferences: the elasticity of intertemporal substitution in individual consumption

$$\epsilon_x^i(x_i) = -u_i'(x_i)/u_i''(x_i)x_i > 0 \tag{19}$$

and the elasticity of the individual labor supply with respect to the wage

$$\epsilon_l^i(l_i) = \frac{dl_i}{dw}\frac{w}{l_i} = -v_i'(\bar{l} - l_i)/v_i''(\bar{l} - l_i)l_i > 0 \tag{20}$$

The following Lemma characterizes the effect on the aggregate steady state of an increase of some θ_i, which necessarily implies a decrease of some other θ_j, everything else being equal.

Lemma 3. *Let Assumptions 1 and 2 hold. Then* $\partial\ell^*/\partial\theta_i > 0$ *if and only if*

$$\frac{\epsilon_l^j(l_j)}{\epsilon_x^j(x_j)} > \frac{\epsilon_l^i(l_i)}{\epsilon_x^i(x_i)} \tag{21}$$

Moreover, $\partial x^*/\partial\theta_i = T(\kappa^*, \mu\kappa^*, 1)\partial\ell^*/\partial\theta_i$.

The intuition of Lemma 3 is the following. At the individual level, an increase of θ_i implies an increase of consumption $x_i(\theta)$ and a decrease of labor $l_i(\theta)$ for agent i, and a decrease of consumption $x_j(\theta)$ and an increase of labor $l_j(\theta)$ for agent j.[13] Therefore, at the aggregate level, an increase of θ_i generates larger aggregate consumption x^* and labor ℓ^* if agent i's reaction is relatively stronger with respect to consumption and relatively weaker with respect to leisure than agent j's reaction. This property is obtained under condition (21).

Building on Theorem 1 and Lemma 3, we need to introduce conditions that characterize the relationship between the aggregate steady state and the distribution of individual wealth. Consider the bounds Λ_j and Γ_j, $j = c, f$, defined by (34) and (35) in Appendix 9.3. Let us denote $\Upsilon_j = \Gamma_j T^* \left[\rho'(x^*) - \Lambda_j\right]$ and $(\Psi, \Phi) \in \{(\Lambda_c, \Upsilon_c), (\Lambda_f, \Upsilon_f)\}$.

Definition 5. *Let* $(\Psi, \Phi) \in \mathbb{R}^2$ *be given. Agents* i *and* j *are said to be elasticity-ordered for* (Ψ, Φ) *if and only if one of the following sets of conditions hold:*

i) $\epsilon_l^j(l_j)/\epsilon_x^j(x_j) < \epsilon_l^i(l_i)/\epsilon_x^i(x_i)$, $\rho'(x^*) < \Psi$ *and* $\gamma'(\ell^*) > \Phi$,
ii) $\epsilon_l^j(l_j)/\epsilon_x^j(x_j) > \epsilon_l^i(l_i)/\epsilon_x^i(x_i)$, $\rho'(x^*) > \Psi$ *and* $\gamma'(\ell^*) < \Phi$.

Conditions i) and ii) are motivated by the result in Lemma 3. They ensure that the aggregate labor supply is a decreasing or an increasing function of the share of capital θ_i owned by agent i and thus require different properties for the slopes of $\rho(x^*)$ and $\gamma(\ell^*)$.

Definition 5 introduces crucial conditions that can be satisfied in two different circumstances: either when agents have homogeneous preferences

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[13] See Appendix 9.1 for a proof of this statement.

characterized by non-linear $\rho_i(x_i)$ and $\gamma_i(l_i)$ (*i.e.*, non constant elasticities $\epsilon_x^i(x_i)$ and $\epsilon_l^i(l_i)$), or when agents have heterogeneous preferences. This is related to the fact that the non-linearity of the *social* absolute risk tolerance indices $\rho(x^*)$ and $\gamma(\ell^*)$ does not require the non-linearity of the *individual* absolute risk tolerance indices.[14]

The following Theorem is the most general result of the paper. It shows that under the stated conditions a sufficiently high level of wealth inequality leads to endogenous business cycle fluctuations in a neighborhood of the steady state.

Theorem 3. *Let Assumptions 1 and 2 hold. Let* $\prec_I^{(\Psi,\Phi)}$ *be the restriction of* $\prec_I$ *to the pairs of elasticity-ordered agents for* (Ψ,Φ) *according to Definition 5. Then there exists a distribution* θ^0 *such that one of the following cases holds:*

1 - If $(\Psi,\Phi) = (\Lambda_c, \Upsilon_c)$ *and Assumption 3 is satisfied, the steady state is saddle-point stable with monotone convergence for any economy E such that* $\theta^E \prec_I^{(\Psi,\Phi)} \theta^0$ *and is saddle-point stable with oscillations otherwise.*

2 - If $(\Psi,\Phi) = (\Lambda_f, \Upsilon_f)$ *and Assumption 4 is satisfied, the steady state is saddle-point stable with oscillating convergence for any economy E such that* $\theta^E \prec_I^{(\Psi,\Phi)} \theta^0$ *and is unstable with oscillating divergence otherwise. Moreover, there generically exist saddle-point stable period-two cycles for any economy E characterized by a distribution* θ^E *in a right neighborhood of* θ^0.

The intuition of this result is as follows. As labor is endogenous, the aggregate steady state (x^*, ℓ^*) directly depends on the distribution of initial shares of capital $\theta = (\theta_i)_{i=1}^n$. Depending on whether (x^*, ℓ^*) is increasing or decreasing in some given share θ_i, Definition 5 provides restrictions on the slopes of $\rho(x)$ and $\gamma(\ell)$ to generate the basic mechanism at the core of Theorem 3: increasing the dispersion of wealth through a bilateral transfer translates into a social utility function with less curvature with respect to consumption (higher $\rho(x^*)$) and larger curvature with respect to labor (lower $\gamma(\ell^*)$).

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[14] Building on Wilson [40], Hara *et al.* [24] show that the social absolute risk tolerance index for consumption is non-linear even when the individual indices are linear provided there is heterogeneity in utility functions among some agents.

Theorem 3 is particularly useful to analyze two specific configurations. The following two subsections are dedicated to this investigation.

7.1 Heterogeneous CRRA preferences

We consider the case of heterogeneous preferences with a CRRA utility function such that

$$U_i(x^i, \mathcal{L}^i) = x_{it}^{1-\sigma_i}/(1-\sigma_i) - (\bar{l} - \mathcal{L}_{it})^{1+\gamma_i}/(1+\gamma_i) \tag{22}$$

with $\sigma_i \geq 0$ and $\gamma_i \geq 0$. Consequently, $\epsilon_x^i(x_i) = 1/\sigma_i$ and $\epsilon_l^i(l_i) = 1/\gamma_i$, with $l_i = \bar{l} - \mathcal{L}_i$. Consider a transfer from agent j to agent i with $\theta_i > \theta_j$.[15] It follows that the conditions on the elasticities in Definition 5 introduce a correlation between the heterogeneity of wealth and the heterogeneity of preferences.

In case ii), the restriction $\epsilon_l^j(l_j)/\epsilon_x^j(x_j) > \epsilon_l^i(l_i)/\epsilon_x^i(x_i)$ becomes $\sigma_i/\gamma_i < \sigma_j/\gamma_j$. In particular, this is satisfied if the richer agent i, who receives a higher share θ_i, is characterized by a higher risk-tolerance with respect to consumption ($\sigma_i < \sigma_j$) but a lower risk tolerance with respect to labor ($\gamma_i > \gamma_j$) than the poorer agent j. Note also that since the individual absolute risk tolerance indices are increasing functions, the aggregate absolute risk tolerance indices are also increasing functions of consumption and labor:[16]

$$\rho'(x) = \textstyle\sum_{i=1}^n \frac{\partial x_i}{\partial x} \frac{1}{\sigma_i} > 0, \quad \gamma'(\ell) = \sum_{i=1}^n \frac{\partial l_i}{\partial \ell} \frac{1}{\gamma_i} > 0$$

For a given set of technologies leading to the bounds $(\Lambda_c, \Lambda_f, \Upsilon_c, \Upsilon_f)$, sufficiently low values for the σ_i's, and sufficiently large values for the γ_i's, $i = 1, \ldots, n$, allow conditions ii) of Definition 5 to be satisfied. In this case, an increase of wealth inequality generates endogenous fluctuations. This configuration is similar to the one covered by Theorem 2 with homogeneous agents in which the individual elasticity of intertemporal substitution in consumption is required to be large enough.

However, the consideration of heterogeneous preferences also allows to get a positive correlation between wealth inequality and output fluctuations under quite different conditions. If $\sigma_i/\gamma_i > \sigma_j/\gamma_j$, i.e., if the richer agent i is characterized by a lower risk-tolerance with respect to consumption

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[15] Note that several transfers can be considered consecutively.

[16] Using Theorem 1 and Remark 1, and applying the same methodology as in Appendix 9.1, it is easy to show that $\partial x_i/\partial x > 0$ and $\partial l_i/\partial \ell > 0$ for any $i = 1, \ldots, n$.

($\sigma_i > \sigma_j$) but a higher risk tolerance with respect to labor ($\gamma_i < \gamma_j$) than the poorer agent j, conditions i) of Definition 5 are satisfied for sufficiently large values for the σ_i's but without any restriction on the γ_i's.[17] In this case, an increase in wealth inequality generates endogenous fluctuations when the individual elasticities of intertemporal substitution in consumption are sufficiently low.

7.2 Homogeneous preferences with non-linear individual absolute risk tolerance index

We consider the case of homogeneous preferences characterized by non-linear individual absolute risk tolerance indices. This is a very large class of preferences (in fact it is even an open and dense set). However, as what matters for the analysis is the slope and curvature of the absolute risk tolerance indices, we restrict the analysis to preferences leading to indices ρ_i and γ_i with a dominant power term. Formally, we focus on the specific class of preferences generating the following risk tolerances:

$$\rho_i(x_i) = \frac{x_i^{\varphi}}{\sigma} \text{ and } \gamma_i(l_i) = \frac{(\bar{l}-\mathcal{L}_i)^{\nu}}{\gamma} = \frac{l_i^{\nu}}{\gamma}$$

with $\sigma, \varphi, \gamma, \nu > 0$, $\varphi \neq 1$, $\nu \neq 1$ and $\mathcal{L}_i = \bar{l} - l_i$.[18]

The restriction $\epsilon_l^j(l_j)/\epsilon_x^j(x_j) \lesseqgtr \epsilon_l^i(l_i)/\epsilon_x^i(x_i)$ becomes in this case $(x_i/x_j)^{\varphi-1} \lesseqgtr (l_i/l_j)^{\nu-1}$. As $\theta_i > \theta_j$, we get from the normality of consumption and leisure that $x_i > x_j$ and $l_i < l_j$. Therefore, when $\varphi > 1$ and $\nu > 1$ case ii) of Definition 5 applies as we have $\epsilon_l^j(l_j)/\epsilon_x^j(x_j) > \epsilon_l^i(l_i)/\epsilon_x^i(x_i)$. Moreover, we easily compute

$$\rho'(x) = \sum_{i=1}^{n} \frac{\partial x_i}{\partial x} \frac{\varphi x_i^{\varphi-1}}{\sigma} > 0, \quad \gamma'(\ell) = \sum_{i=1}^{n} \frac{\partial l_i}{\partial \ell} \frac{\nu l_i^{\nu-1}}{\gamma} > 0$$

For a given set of technologies leading to the bounds $(\Lambda_c, \Lambda_f, \Upsilon_c, \Upsilon_f)$, sufficiently low values for the σ_i's, and sufficiently large values for the γ_i's, $i = 1, \ldots, n$, allow conditions ii) of Definition 5 to be satisfied. Consequently, when individual risk tolerances are convex functions, an increase of

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[17] Indeed, $\gamma'(\ell) > 0$ while $\Upsilon_i < 0$ as soon as $\rho'(x) < \Lambda_i$, $i = c, f$.

[18] This class of preferences is large and represents a minimal deviation from the CRRA formulation (see Gollier [22]). Choosing values for φ and ν close enough to 1 allows individual absolute risk tolerance indices to be considered arbitarily close to the linear formulation characterizing CRRA utility functions.

wealth inequality generates endogenous fluctuations if the individual elasticities of intertemporal substitution in consumption are large enough but the individual elasticities of labor are low enough.

However, a positive correlation between wealth inequality and output fluctuations can also be obtained when $\varphi < 1$ and $\nu < 1$, i.e., with concave individual risk tolerances. Conditions i) of Definition 5 are indeed satisfied with sufficiently large values for the σ_i's but without any restriction on the γ_i's. Under concave risk tolerance indices, an increase of wealth inequality still generates endogenous fluctuations provided the individual elasticities of intertemporal substitution in consumption are low enough.

7.3 The empirical plausibility of the configurations

The analysis of these two specific configurations shows that the results of Theorem 3 do not rest on the properties of the curvature of the individual risk tolerance indices. Indeed, contrary to Ghiglino and Venditti [21], a positive correlation between wealth inequality and output fluctuations can be obtained with linear, convex or concave $\rho_i(x_i)$ and $\gamma_i(l_i)$.

We now evaluate the plausibility of the conditions stated in Theorem 3, and discussed within the two previous configurations, in light of the available empirical findings. First, the restriction on the capital intensity difference across sectors is compatible with recent empirical evidence. In particular, building on aggregate Input-Output tables, Takahashi et al. [37] have shown that over the last 30 years the OECD countries have been characterized by a *consumption good sector that is more capital-intensive than the investment good sector.*[19]

Second, Guiso and Paiella [23] provide an empirical investigation of absolute risk tolerance for consumption. Household survey data are used to construct a direct measure of absolute risk tolerance based on the maximum price a consumer is willing to pay to buy a risky security. The analysis shows that *risk tolerance is an increasing function of endowment.*

Third, concerning the elasticity of the individual labor supply with respect to the wage, Rogerson and Wallenius [33] have shown that the corresponding elasticities at the macroeconomic level are virtually unrelated

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[19] Baxter [2] obtains similar results.

to the micro elasticities, and are much larger than expected. The reason for this discrepancy is attributed to the role of the participation decision of women and the extent of early retirement. In particular, the empirical estimation at the micro level indicate that the wage elasticity of labor belongs to $(0, 0.5)$ for men and to $(0.5, 1.25)$ for women (see Blundell and MaCurdy [10]) while according to Rogerson and Wallenius [33] the corresponding "macroeconomic" *elasticity of the labor supply with respect to the wage are is the range of* $2.25 - 3.0$.

Finally, while many standard RBC models usually assume a unitary elasticity of intertemporal substitution in consumption, recent empirical estimates provide divergent views. On the one hand, Campbell [14] and Kocherlakota [27] suggest the interval $0.2 - 0.8$. More recently Vissing-Jorgensen [39] has partly confirmed such findings showing that the estimates of this elasticity are around $0.3 - 0.4$ for stockholders and around $0.8 - 1$ for bondholders and are larger for households with larger asset holdings within these two groups. On the other hand, Mulligan [29] repeatedly obtains estimates of one and above, i.e. around $1.1 - 1.3$, using different estimation methods. Overall the estimates for the *elasticity of intertemporal substitution in consumption is in the range* $0.2 - 1.3$.

The two types of restrictions required by Theorem 3 for a positive correlation between wealth inequality and output fluctuations are in lines with these empirical findings. Indeed, first consider Case ii) of Definition 5. In this case, individual elasticity of the labor supply is low and compatible with micro estimates. When agents have heterogeneous CRRA preferences, an increase of wealth inequality generates output fluctuations if the σ_i's are low enough and satisfy $\sigma_i/\gamma_i < \sigma_j/\gamma_j$. On the other hand, when homogeneous preferences characterized by non-linear risk tolerance indices are considered, the same result holds if the σ_i's are low enough and $\rho_i(x_i)$ and $\gamma_i(l_i)$ are convex functions. In both these configurations, the focus on large enough individual elasticities of intertemporal substitution in consumption is supported by the findings of Mulligan [29].

Case ii) of Definition 5, i.e., low individual elasticities of intertemporal substitution in consumption, is compatible with the estimates of Campbell [14] and Kocherlakota [27]. When agents have heterogeneous CRRA preferences, an increase of wealth inequality generates output fluctuations if $\sigma_i/\gamma_i > \sigma_j/\gamma_j$. When homogeneous preferences characterized by non-linear risk tolerance indices are considered, the same result holds if $\rho_i(x_i)$ and $\gamma_i(l_i)$ are concave functions. In both configurations, the conclusions hold without any restriction on the individual elasticities of labor supply and thus can be compatible with the findings of Rogerson and Wallenius [33].

Remark 4. *Note finally that under the assumptions in Theorem 3, if we consider a benchmark economy characterized by endogenous fluctuations, a rise in inequality leads to an increase in the magnitude of fluctuations.*

8 Conclusion

We consider a two-sector optimal growth model with endogenous labor and heterogeneous agents with respect to their initial capital shares and their individual endowments. First, we prove that a positive correlation between wealth inequality and output fluctuations arises when the individual wealth is Pareto distributed and agents have homogeneous CRRA preferences. Second, we extend this conclusion to general distributions of individual wealth. These results are applied to economies in which the individual absolute risk tolerance indices for consumption and labor are linear provided there is some degree of preference heterogeneity across agents, and to economies in which the individual absolute risk tolerance indices are non-linear and agents have homogeneous preferences.

The model of this paper can be extended to a multi-sector optimal growth framework with at least two investment goods. In such a case, in addition to period-two cycles, quasi-periodic fluctuations can be obtained as a result of increased inequality.[20] In a second extension, the set of utility functions can be enlarged as to include habit formation, status effects and other types of consumption externalities. The introduction of financial constraints can also reveal to be interesting. All this is left for future research.

It should be finally noted that there is currently a renewal of interest in the possibility that macroeconomic fluctuations could affect the distribution of wealth, an issue we do not consider in the paper. For instance,

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[20] See Benhabib and Nishimura [5] and Venditti [38] for the existence of Hopf bifurcation in multi-sector optimal growth models.

Caroli and García-Peñalosa [15] show that higher volatility increases income inequality if agents with different endowments have different attitudes towards risk,[21] while García-Peñalosa and Turnovsky [18] provide a similar conclusion through the effect of greater production uncertainty.[22] Our model could also be extended along these lines through the introduction of stochastic technological disturbances (see Benhabib and Nishimura [8]).

9 Appendix

9.1 Proof of Theorem 1

Denoting $\kappa = k/\ell$, we derive from (13)-(14) and the first order conditions associated with program (4) that an aggregate steady state can be defined as a pair (κ^*, ℓ^*) solution of the following equations

$$-\tfrac{T_1(\kappa,\mu\kappa,1)}{T_2(\kappa,\mu\kappa,1)} = f_1^1(k_1(\kappa,\mu\kappa,1), l_1(\kappa,\mu\kappa,1)) = \delta[1-\delta(1-\mu)]^{-1} \equiv \delta\vartheta \quad (23)$$

$$u'(\ell T(\kappa,\mu\kappa,1))T_3(\kappa,\mu\kappa,1) - v'(\ell) = 0 \quad (24)$$

Consider in a first step equation (23). The proof of Theorem 3.1 in Becker and Tsyganov [4] applies so that there exists a unique κ^* solution of (23). Consider in a second step equation (24) evaluated at κ^*. We get:

$$T_3(\kappa^*,\mu\kappa^*,1) = \tfrac{v'(\ell)}{u'(\ell T(\kappa^*,\mu\kappa^*,1))} \equiv \varphi(\ell)$$

The function $\varphi(\ell)$ is defined over $(0,\bar{\ell})$ and satisfies

$$\varphi'(\ell) = \tfrac{u'(x)v''(\ell) - u''(x)v'(\ell)T}{u'(x)^2} > 0$$

This property together with the boundary conditions in Lemma 1 finally guarantee the existence and uniqueness of a solution $\ell^* \in (0,\bar{\ell})$ of (24).

Let us now consider the first order conditions corresponding to the individual maximization of the intertemporal utility function (1) subject to the intertemporal budget constraint (2):

$$\delta^t u_i'(x_{it}) = \pi_i R_t \quad (25)$$

$$\delta^t v_i'(\bar{l} - l_{it}) = \pi_i R_t w_t \quad (26)$$

$$\sum_{t=0}^{\infty} R_t x_{it} = \sum_{t=0}^{\infty} R_t(\omega_i + w_t l_{it}) + \theta_i r_0 k_0 \quad (27)$$

---

[21] See also Cecchi and García-Peñalosa [16].

[22] See also García-Peñalosa and Turnovsky [17].

$\forall t \geq 0$ and $i = 1, \ldots, n$, where π_i is the Lagrangian multiplier associated with the intertemporal budget constraint (2). From (6) we conclude that

$$1 + d_t = \frac{r_t + (1-\mu)p_t}{p_{t-1}} = -\frac{T_1(k_t, y_t) - (1-\mu)T_2(k_t, y_t)}{T_2(k_{t-1}, y_{t-1})}$$

for any $t \geq 1$ and $1 + d_0 = r_0/p_{-1}$ for $t = 0$. The Euler equation (13) evaluated at a steady state $x_{it} = x_i^*$ gives $1 + d^* = \delta^{-1}$ and thus $R_t = \delta^t$. Recall that from (6) we also get $T_1^* = r^*$, $T_2^* = -p^*$ and $w^* = T_3^*$. The intertemporal budget constraint (27) evaluated along the stationary path with $k_t = k^*$ for all $t \geq 0$ and $p_{-1} = p^*$ becomes $x_i^* = \omega_i + w^* l_i^* + (1-\delta)\theta_i p^* k^*/\delta$, with $p^* = \delta \vartheta r^*$. We then get

$$x_i^* = \omega_i + w^* l_i^* + (1-\delta)\vartheta r^* \kappa^* \ell^* \theta_i \tag{28}$$

with $\kappa^* = k^*/\ell^*$. Using (25) and (26), we obtain

$$u_i'(\omega_i + w^* l_i^* + (1-\delta)\vartheta r^* \kappa^* \ell^* \theta_i) w^* = v_i'(\bar{l} - l_i^*) \tag{29}$$

Assumption 1 implies $\lim_{l_i \to \bar{l}} v_i'(\bar{l} - l_i) = +\infty > u_i'(\omega_i + w^* \bar{l} + (1-\delta)\vartheta r^* \kappa^* \ell^* \theta_i) w^*$. Therefore, if $v_i'(\bar{l}) < u_i'(\omega_i + (1-\delta)\vartheta r^* \kappa^* \ell^* \theta_i)$ for any $i = 1, \ldots, n$, there exist unique interior steady state values for all individual consumptions x_i^* and labor supplies l_i^* solutions of equations (28) and (29). Note that since equation (28) depends on θ_i and ω_i, and $\ell^* = \sum_{i=1}^n l_i^*$, we conclude that x_i^* and l_i^* are functions of the distribution of capital shares $\theta = (\theta_i)_{i=1}^n$ and the distribution of individual endowments $\omega = (\omega_i)_{i=1}^n$, namely $x_i^*(\theta, \omega)$ and $l_i^*(\theta, \omega)$. For all $i = 1, \ldots, n$, consider finally equations (28) and (29) expressed as follows:

$$\begin{aligned} x_i^* - \omega_i - w^* l_i^* - (1-\delta)\vartheta r^* \kappa^* \left(\sum_{i=1}^n l_i^* \right) \theta_i &= 0 \\ u_i'(x_i^*) w^* - v_i'(\bar{l} - l_i^*) &= 0 \end{aligned} \tag{30}$$

with $\theta_j = 1 - \sum_{i=1, i \neq j}^n \theta_i$ and $\omega_j = \bar{\omega} - \sum_{i=1, i \neq j}^n \omega_i$. Applying the implicit function theorem, we now show that $x_i^*(\theta, \omega)$ and $l_i^*(\theta, \omega)$ are C^1-functions. The $2n \times 2n$ Jacobian matrix of (30) with respect to $(x_i^*, l_i^*)_{i=1}^n$ is

$$J_{(x_i, l_i)} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

with $J_{11} = I_{n \times n}$,

$$J_{21} = w^* \begin{pmatrix} u_1''(x_1^*) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & u_n''(x_n^*) \end{pmatrix}, J_{22} = \begin{pmatrix} v_1''(\bar{l} - l_1^*) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & v_n''(\bar{l} - l_n^*) \end{pmatrix}$$

and

$$J_{12} = -T^* I_{n\times n} + (1-\delta)\vartheta r^* \kappa^*$$

$$\times \begin{pmatrix} 1-\theta_1 & \theta_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \theta_1 \\ \theta_2 & 1-\theta_2 & \theta_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \theta_2 \\ \vdots & & \ddots & \vdots & \vdots & & & & \vdots \\ \theta_{j-1} & \cdots & \theta_{j-1} & 1-\theta_{j-1} & \theta_{j-1} & \cdots & \cdots & \cdots & \theta_{j-1} \\ 1-\sum\limits_{\substack{i=1\\ i\neq j}}^{n}\theta_i & \cdots & \cdots & 1-\sum\limits_{\substack{i=1\\ i\neq j}}^{n}\theta_i & \sum\limits_{\substack{i=1\\ i\neq j}}^{n}\theta_i & 1-\sum\limits_{\substack{i=1\\ i\neq j}}^{n}\theta_i & \cdots & \cdots & 1-\sum\limits_{\substack{i=1\\ i\neq j}}^{n}\theta_i \\ \theta_{j+1} & \cdots & \cdots & \cdots & \theta_{j+1} & 1-\theta_{j+1} & \theta_{j+1} & \cdots & \theta_{j+1} \\ \vdots & & & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & & & \vdots & \vdots & & & \ddots & \vdots \\ \theta_n & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \theta_n & 1-\theta_n \end{pmatrix}$$

Consider the matrix $B = J_{22} - J_{21}J_{12}$. Tedious computations available upon request show that $|B| \neq 0$ with $sign|B| = (-1)^n$ and $B^{-1} = |B|^{-1}\left[b_{ij}\right]_{i,j=1}^{n}$ with $sign\ b_{ii} = (-1)^{n-1}$ and $sign\ b_{ij} = (-1)^n$ for $i \neq j$. Therefore, the Jacobian matrix $J_{(x_i,l_i)}$ admits an inverse such that

$$J_{(x_i,l_i)}^{-1} = \begin{pmatrix} I_{n\times n} + J_{12}B^{-1}J_{21} & -J_{12}B^{-1} \\ -B^{-1}J_{21} & B^{-1} \end{pmatrix}$$

Tedious computations available upon request also show that

$$I_{n\times n} + J_{12}B^{-1}J_{21} = |B|^{-1}\left[|B|I_{n\times n} + J_{12}\left[b_{ij}\right]J_{21}\right] \equiv |B|^{-1}\left[c_{ij}\right]_{i,j=1}^{n}$$

with $sign\ c_{ii} = (-1)^n$ and $sign\ c_{ij} = (-1)^{n-1}$ for $i \neq j$. The $2n \times (n-1)$ Jacobian matrix of (30) with respect to $\theta = (\theta_i)_{i=1}^{n}$ with $\theta_j = 1 - \sum_{i=1, i\neq j}^{n}\theta_i$ is

$$J_\theta = (1-\delta)\vartheta r^* \kappa^* \ell^* \left(\begin{array}{ccccccccc} -1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & \vdots & \vdots & & & & \vdots \\ \vdots & & \ddots & 0 & \vdots & & & & \vdots \\ 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & 1 \\ 0 & \cdots & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & & & \vdots & \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & -1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{array}\right) \begin{array}{l} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} j-1 \\ \\ \text{line } j \\ \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} n-j \\ \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} n \end{array}$$

Similarly, the $2n\times(n-1)$ Jacobian matrix of (30) with respect to $\omega=(\omega_i)_{i=1}^n$ with $\omega_j=\bar{\omega}-\sum_{i=1,i\neq j}^n \omega_i$ is

$$J_\omega=\left(\begin{array}{ccccccccc}
-1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & & \vdots & \vdots & & & & \vdots \\
\vdots & & \ddots & 0 & \vdots & & & & \vdots \\
0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 \\
\vdots & & & \vdots & \vdots & & \ddots & & \vdots \\
\vdots & & & \vdots & \vdots & & & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & -1 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & & & & & & & & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{array}\right)
\begin{array}{l}
\\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \quad j-1 \\
\\ \text{line } j \\
\\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \quad n-j \\
\\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \quad n
\end{array}$$

We conclude from the implicit function theorem that $x_i^*(\theta,\omega)$ and $l_i^*(\theta,\omega)$ are C^1-functions with

$$\left[\frac{\partial x_i}{\partial \theta_j}\right]_{n\times(n-1)} = -\frac{(1-\delta)\vartheta r^*\kappa^*\ell^*}{|B|}\,[c_{ij}]\left(\begin{array}{cccc}
-1 & 0 & \cdots & 0 \\
0 & \ddots & & \vdots \\
\vdots & & \ddots & 0 \\
0 & \cdots & 0 & -1 \\
1 & \cdots & \cdots & 1 \\
 & & &
\end{array}\right)$$

and

$$\left[\frac{\partial l_i}{\partial \theta_j}\right]_{n\times(n-1)} = \frac{(1-\delta)\vartheta r^*\kappa^* w^*\ell^*}{|B|}\,[b_{ij}]$$

$$\times\left(\begin{array}{ccccccccc}
-u_1''(x_1^*) & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & & \vdots & \vdots & & & & \vdots \\
\vdots & & \ddots & 0 & \vdots & & & & \vdots \\
0 & \cdots & 0 & -u_{j-1}''(x_{j-1}^*) & 0 & \cdots & \cdots & \cdots & 0 \\
u_j''(x_j^*) & \cdots & \cdots & u_j''(x_j^*) & u_j''(x_j^*) & u_j''(x_j^*) & \cdots & \cdots & u_j''(x_j^*) \\
0 & \cdots & \cdots & 0 & 0 & -u_{j+1}''(x_{j+1}^*) & 0 & \cdots & 0 \\
\vdots & & & \vdots & \vdots & & \ddots & & \vdots \\
\vdots & & & \vdots & \vdots & & & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & -u_n''(x_n^*)
\end{array}\right)$$

It follows that

$$\begin{aligned}
\frac{\partial x_i^*(\theta)}{\partial \theta_i} &= -\tfrac{(1-\delta)\vartheta r^* \kappa^* \ell^* (c_{ij}-c_{ii})}{|B|} > 0 \\
\frac{\partial x_j^*(\theta)}{\partial \theta_i} &= -\tfrac{(1-\delta)\vartheta r^* \kappa^* \ell^* (c_{jj}-c_{ji})}{|B|} < 0 \\
\frac{\partial l_i^*(\theta)}{\partial \theta_i} &= \tfrac{(1-\delta)\vartheta r^* \kappa^* w^* \ell^* (b_{ij} u_j''(x_j^*)-b_{ii} u_i''(x_i^*))}{|B|} < 0 \\
\frac{\partial l_j^*(\theta)}{\partial \theta_i} &= \tfrac{(1-\delta)\vartheta r^* \kappa^* w^* \ell^* (b_{jj} u_j''(x_j^*)-b_{ji} u_i''(x_i^*))}{|B|} > 0
\end{aligned} \tag{31}$$

Obviously, the same type of results are obtained for the partial derivatives of $x_i^*(\theta,\omega)$ and $l_i^*(\theta,\omega)$ with respect to ω_j.

□

9.2 Proof of Proposition 1

In a one-sector economy with heterogeneous agents, Kehoe *et al.* [26] show that the welfare weights are continuous functions of the initial capital stock. This property holds because the value function of the planner's problem (10) is C^2. However, in a multi-sector economy Santos [35] shows that the main sufficient condition to get such a property is to assume strong concavity of the indirect utility function $\mathcal{V}(k_t, k_{t+1})$ (see Assumption B and Theorem 2.2 in Santos [35]). On a compact set, a C^2 function $\mathcal{V}(k_t, k_{t+1})$ is strongly concave if its Hessian matrix is non-singular and negative-definite. In other words, the smallest eigenvalue of the Hessian matrix in absolute value needs to be strictly positive over the domain of definition of $\mathcal{V}(k_t, k_{t+1})$. In our model, the indirect social utility function also depends on ℓ_t:

$$V(k_t, k_{t+1}, \ell_t) = u(\bar{\omega} + T(k_t, k_{t+1} - (1-\mu)k_t, \ell_t)) - v(\ell_t)$$

with ℓ_t a solution of equation (14):

$$u'(\bar{\omega} + T(k_t, k_{t+1} - (1-\mu)k_t, \ell_t))w_t - v'(\ell_t) \equiv \phi(k_t, k_{t+1}, \ell_t) = 0$$

Since under Assumptions 1 and 2

$$\phi_3(k_t, k_{t+1}, \ell_t) = u''(x_t)w_t^2 + u'(x_t)T_{33} - v''(\ell_t) < 0$$

we derive from the implicit function theorem that $\ell_t = \psi(k_t, k_{t+1})$ with $\psi(.)$ a C^1-function such that

$$\begin{aligned}
\psi_1(k_t, k_{t+1}) &= -\tfrac{u''(x_t)(r_t+(1-\mu)p_t)w_t + u'(x_t)(T_{31}-(1-\mu)T_{32})}{u''(x_t)w_t^2 + u'(x_t)T_{33} - v''(\ell_t)} \\
\psi_2(k_t, k_{t+1}) &= -\tfrac{u'(x_t)T_{32} - u''(x_t)p_t w_t}{u''(x_t)w_t^2 + u'(x_t)T_{33} - v''(\ell_t)}
\end{aligned}$$

Then we obtain

$$\mathcal{V}(k_t, k_{t+1}) = u(\bar{\omega} + T(k_t, k_{t+1} - (1-\mu)k_t, \psi(k_t, k_{t+1}))) - v(\psi(k_t, k_{t+1}))$$

Now consider $T(k_t, y_t, \ell_t)$ with $y_t = k_{t+1} - (1-\mu)k_t$ and $\ell_t = \psi(k_t, k_{t+1})$. Proceeding in a similar way as for the set of admissible paths $\tilde{\mathcal{D}}$, we derive that $T(k_t, y_t, \ell_t) = 0$ if and only if $k_{t+1} = h(k_t)$. Assumption 2 implies that there exists $\hat{k} > 0$ such that $h(k_t) > k_t$ when $k_t < \hat{k}$ while $h(k_t) < k_t$ when $k_t > \hat{k}$. As a result, $\mathcal{V}(k_t, k_{t+1})$ is defined over the compact, convex set

$$\hat{\mathcal{D}} = \Big\{(k_t, k_{t+1}) \in \mathbb{R}^2_+ | 0 \leq k_t \leq \hat{k},\ (1-\mu)k_t \leq k_{t+1} \leq h(k_t)\Big\}$$

We know that T is homogeneous of degree 1 so that its Hessian matrix $H_T(k_t, k_{t+1})$ is singular for any $(k_t, k_{t+1}) \in \hat{\mathcal{D}}$. As shown in Benhabib and Nishimura [7] and Bosi *et al.* [12], we have

$$\begin{aligned}
T_{12}(k, y, \ell) &= -T_{11}(k, y, \ell) b(k, y, \ell) \\
T_{22}(k, y, \ell) &= T_{11}(k, y, \ell) b(k, y, \ell)^2 \\
T_{13}(k, y, \ell) &= -T_{11}(k, y, \ell) a(k, y, \ell) \\
T_{23}(k, y, \ell) &= T_{11}(k, y, \ell) a(k, y, \ell) b(k, y, \ell) \\
T_{33}(k, y, \ell) &= T_{11}(k, y, \ell) a(k, y, \ell) b(k, y, \ell)^2
\end{aligned} \tag{32}$$

where

$$b(k, y, \ell) = \tfrac{l^0}{T}\left(\tfrac{k^1}{l^1} - \tfrac{k^0}{l^0}\right) \tag{33}$$

and $a(k, y, \ell) = k^0/l^0 > 0$. Tedious but straightforward computations then yield to the determinant of the Hessian matrix of $\mathcal{V}$, which is equal to

$$|H_{\mathcal{V}}| = -\tfrac{u'(x_t)u''(x_t)v''(\ell_t)T_{11}(b_t r_t - p_t)^2}{u''(x_t)w_t^2 + u'(x_t)T_{33} - v''(\ell_t)} \geq 0$$

with $b_t = b(k_t, k_{t+1} - (1-\mu)k_t, \psi(k_t, k_{t+1}))$. Under Assumptions 1-2, Lemma 1 implies that $|H_{\mathcal{V}}| > 0$ if over the interior of the set $\hat{\mathcal{D}}$ we have $b_t r_t - p_t \neq 0$ or equivalently, $T_2(k_t, k_{t+1} - (1-\mu)k_t, \ell_t) + b(k_t, k_{t+1} - (1-\mu)k_t, \ell_t)T_1(k_t, k_{t+1} - (1-\mu)k_t, \ell_t) \neq 0$. This property also implies that the value function of the planner's problem (10) is C^2. □

9.3 Proof of Propositions 3 and 4

In Proposition 3, assume $b \in (-1/(1-\mu), -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0)$ and consider the critical values:

$$
\begin{aligned}
\rho_c &= -\tfrac{\delta\vartheta^2\varepsilon_{ck}x^*}{b[1+(1-\mu)b]\varepsilon_{rk}} \equiv \Lambda_c x^* \\
\gamma_c &= -\frac{\frac{1-s}{s}\left(\frac{1}{1-\mu b}\right)^2\left[(1-\delta)\vartheta+\frac{1-s}{s}\right]b[1+(1-\mu)b]}{\left[\frac{1-s}{s}\frac{b}{1-\mu b}+\delta\vartheta\right]\left[\frac{1-s}{s}\frac{1+(1-\mu)b}{1-\mu b}+\vartheta\right]}\left[\rho(x^*)-\rho_c\right] \equiv \Gamma_c\left[\rho(x^*)-\rho_c\right]
\end{aligned}
\tag{34}
$$

with $\vartheta = [1-\delta(1-\mu)]^{-1}$, $\rho_c > 0$, and $\gamma_c > 0$ for $\rho(x^*) > \rho_c$. The rest of the proof is adapted from Bosi *et al.* [12].

In Proposition 4, assume $b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu)))$ and consider the critical values:

$$
\begin{aligned}
\rho_f &= -\frac{2\delta(1+\delta)\vartheta^2\varepsilon_{ck}x^*}{[1+(2-\mu)b]\left[\delta+[1+(1-\mu)\delta]b\right]\varepsilon_{rk}} \equiv \Lambda_f x^* \\
\gamma_f &= -\frac{\frac{1-s}{s}\left(\frac{1}{1-\mu b}\right)^2\left[(1-\delta)\vartheta+\frac{1-s}{s}\right][1+(2-\mu)b][\delta+b(1+\delta(1-\mu))]}{\left[\frac{1-s}{s}\frac{1+(2-\mu)b}{1-\mu b}+(1+\delta)\vartheta\right]\left[\frac{1-s}{s}\frac{\delta+b(1+\delta(1-\mu))}{1-\mu b}+2\delta\vartheta\right]}\left[\rho(x^*)-\rho_f\right] \\
&\equiv \Gamma_f\left[\rho(x^*)-\rho_f\right]
\end{aligned}
\tag{35}
$$

with $0 < \rho_c < \rho_f$, and $0 < \gamma_f < \gamma_c$ when $\rho(x^*) > \rho_f$. The rest of the proof is adapted from Bosi *et al.* [12].

$\square$

9.4 Proof of Lemma 2

Existence and uniqueness. We need to show that for any given $n \in \mathbb{N}$ there exists a unique distribution $\Gamma_n(\alpha)$ with probability measure μ_n such that when $\mu(s)n \notin \mathbb{N}$, $\mu_n(s)$ is defined by $|\mu_n(s)-\mu(s)| \leq 1/2n$ with $\mu_n(s)n \in \mathbb{N}$, and when $\mu(s)n \in \mathbb{N}$, $\mu_n(s) = \mu(s)$. Indeed, for any real number $n\mu(s)$ there always exists $n_s \in \mathbb{N}$ such that $n_s \leq n\mu(s) \leq n_s + 1$. This implies that there exists a sequence $\{u_s\}$ such that $|u_s - n\mu(s)| \leq 1/2$ with $u_s = n_s$ or $n_s + 1$ depending on which one is closer to $n\mu(s)$. The sequence $\{u_s\}$ thus constructed exists and is unique. The distribution $\Gamma_n(\alpha)$ is obtained with $\mu_n(s) = u_s/n$.

Limit. Let the measure μ be given. Then the inequality $|\mu_n(s)-\mu(s)| \leq 1/2n$ implies that for any $\epsilon > 0$ and $s \in S$ there exists $N_\epsilon = Int[\frac{1}{2\epsilon}] + 1$ such that for all $n \geq N_\epsilon$ we have $|\mu_n(s)-\mu(s)| \leq \epsilon$. This shows that the sequence of distributions $\Gamma_n(\alpha)$ of measure μ_n converges pointwise in s to the distribution $\Gamma(\alpha)$ of measure μ.

Implementation. Step 1: For a given $\alpha > 1$ and a given pair of sectorial technologies, we need to show that for a given n, there exist a distribution of shares θ with $\sum_{i=1}^n \theta_i = 1$ and a distribution of individual endowments ω

with $\sum_{i=1}^{n}\omega_i = \bar{\omega}$ generating this specific distribution of individual wealth. The given pair of technologies implies the existence of a unique κ^*. As there is a unique consumption good and agents have a common time discount factor, along the steady state the distribution of wealth is identical to the distribution of individual consumption. We have

$$\begin{aligned} x_i(\theta,\omega) &= \omega_i + wl_i(\theta,\omega) + (1-\delta)\vartheta r\kappa^*\ell\theta_i \\ &= \omega_i + wl_i(\theta,\omega) + (1-\delta)\vartheta rk\theta_i \equiv \Omega_i \end{aligned} \tag{36}$$

with δ, ϑ some parameters and w, r that depend on κ^* but are independent of the distributions of shares θ and individual endowment ω. Furthermore, $u_i'(x_i(\theta,\omega))w = v_i'(\bar{l} - l_i(\theta,\omega))$ implies that $l_i(\theta,\omega)$ can be written as a function of $x_i(\theta,\omega)$. Assume that the vector of individual consumption $(x_1, x_2, ..., x_n)$ is given. Then, both $x^* = \sum_{i=1}^{n} x_i$ and $\ell^* \sum_{i=1}^{n} l_i$ are given and can be computed. Therefore, using the fact that $\sum_{i=1}^{n}\theta_i = 1$, $\sum_{i=1}^{n}\omega_i = \bar{\omega}$ and summing over the agents the budget equation (36), we obtain

$$x^* = \bar{\omega} + w\ell^* + (1-\delta)\vartheta r\kappa^*\ell^* \tag{37}$$

and we derive the associated value for $\bar{\omega}$. As equation (36) holds for each agent i, we obtain a system of n equations. We know from the proof of Theorem 1 that the associated Jacobian matrix has rank $n-1$. Therefore, using at least $n-1$ parameters over the $2(n-1)$ given by $(\theta_i, \omega_i)_{i=1}^{n}$, from (36) we can compute for each x_i the corresponding θ_i and ω_i.

Step 2: We now consider a sequence of economies indexed by n. For each n we repeat step 1. □

9.5 Proof of Theorem 2

We first assume that individual wealth is distributed according to an exact distribution $\Gamma(\alpha)$. Implicitly, the set of agents is described by a continuum of mass 1. We will subsequently show that the result holds for $\Gamma_n(\alpha)$ when n is sufficiently large.

When normalized incomes follow exactly a Zeta distribution along the steady state, normalized individual consumptions $\tilde{x}_i = x_i/x_{\min}$ also follow a Zeta distribution. Under a CRRA utility function, the corresponding value of $\rho(x)$ is given by

$$\rho(x) = \sum_{s=1}^{\infty} \frac{1}{\zeta(\alpha)} \frac{1}{s^{\alpha}} \rho(x(s)) = \frac{x_{\min}}{\sigma} \frac{\zeta(\alpha-1)}{\zeta(\alpha)} = \bar{\rho}(\alpha) \tag{38}$$

where $\zeta(\alpha) = \sum_{s=1}^{\infty} s^{-\alpha}$ is the Riemann Zeta function.

Proceeding similarly for risk tolerance to leisure, from the first order condition (14), we derive under a CRRA utility function that leisure is related to consumption as follows

$$l_i = x_i^{-\sigma/\gamma} w^{1/\gamma} \tag{39}$$

Combining (38) and (39) we then get

$$\gamma(\ell) \quad = \quad \frac{x_{\min}^{-\sigma/\gamma} w^{1/\gamma}}{\gamma} \frac{\zeta(\alpha+\frac{\sigma}{\gamma})}{\zeta(\alpha)} \equiv \bar{\gamma}(\alpha) \tag{40}$$

The properties of the Riemann Zeta function imply that $\bar{\rho}(\alpha)$ and $\bar{\gamma}(\alpha)$ are respectively decreasing and increasing in α.

Using Propositions 3 and 4, we first need to show that $\rho(x^*) > \rho_i = \Lambda_i x^*$ for $i = c$ or f, or equivalently

$$\bar{\rho}(\alpha) > \sum_{s=1}^{\infty} \frac{\Lambda_i}{\zeta(\alpha)} \frac{s x_{min}}{s^{\alpha}} = x_{\min} \Lambda_i \frac{\zeta(\alpha-1)}{\zeta(\alpha)} \tag{41}$$

Using (38) we then get

$$\tfrac{1}{\sigma} > \Lambda_i \tag{42}$$

Let us assume that this inequality is satisfied so that $\rho(x^*)$ is always larger than ρ_i. Now, we need to show that $\gamma(\ell^*)$ can cross the critical value $\gamma_i = \Gamma_i \left[\rho(x^*) - \rho_i\right]$ for $i = c$ or f, or equivalently

$$\bar{\gamma}(\alpha) \gtrless x_{\min} \Gamma_i \left[\tfrac{1}{\sigma} - \Lambda_i\right] \tfrac{\zeta(\alpha-1)}{\zeta(\alpha)} \tag{43}$$

Using (40) and (43) we then get

$$\tfrac{\zeta(\alpha+\frac{\sigma}{\gamma})}{\zeta(\alpha-1)} \gtrless x_{\min}^{1+\sigma/\gamma} \tfrac{\gamma \Gamma_i}{w^{1/\gamma}} \left[\tfrac{1}{\sigma} - \Lambda_i\right] \tag{44}$$

The left-hand-side is an increasing function of α taking values on the interval $(0, 1)$. Therefore, the critical value γ_i can be crossed by varying α if and only if the right-hand-side is strictly less than 1. This can be obtained for an open set Ξ of values of σ and γ. Let $\bar{\alpha}$ be the critical value of α such that (43) is satisfied with an equality. In such a case, as inequalities increase when α decreases, the occurrence of endogenous fluctuations is always obtained from an increase of wealth inequality.

We now show that the result holds for $\Gamma_n(\alpha)$ when n is sufficiently large. From Lemma 2 we know that $lim_{n \to +\infty} \mu_n(s) = \frac{s^{-\alpha}}{\zeta(\alpha)}$. Consequently,

$$\lim_{n\to\infty} \frac{\rho_n(x)}{n} = \sum_{s=1}^{\infty} \lim_{n\to\infty} \mu_n(s) \rho(x(s)) = \sum_{s=1}^{\infty} \frac{1}{\zeta(\alpha)} \frac{1}{s^{\alpha}} \rho(x(s)) = \bar{\rho}(\alpha)$$

Similarly, we derive that

$$\lim_{n\to\infty} \frac{\gamma_n(\ell)}{n} = \bar{\gamma}(\alpha)$$

Let $G(\Gamma_n(\alpha)$ be the Gini index associated to the distribution $\Gamma_n(\alpha)$. Because of the continuity of the functions involved we get that

$$\lim_{n\to\infty} G(\Gamma_n(\alpha)) \quad = \quad \tfrac{1}{2\alpha-1}$$

From Lemma 2, we know that $\Gamma_n(\alpha)$ is implementable for α in a neighborhood of $\bar{\alpha}$.

□

9.6 Proof of Lemma 3

As $\ell^*(\theta) = \sum_{i=1}^n l_i^*(\theta)$ and $x^*(\theta) = \sum_{i=1}^n x_i^*(\theta)$ with $l_i^*(\theta)$ and $x_i^*(\theta)$ some C^1-functions, $\ell^*(\theta)$ and $x^*(\theta)$ are C^1-functions. Moreover, we have

$$\frac{\partial \ell^*(\theta)}{\partial \theta_i} = \sum_{k=1}^n \frac{\partial l_k^*(\theta)}{\partial \theta_i}$$

Recalling that $\theta_j = 1 - \sum_{i=1, i\neq j}^n \theta_i$, tedious computations available upon request give from (31) in Appendix 9.1

$$\begin{aligned} \tfrac{\partial \ell^*(\theta)}{\partial \theta_i} \quad &= \quad \tfrac{(1-\delta)\vartheta r^* \kappa^* w^* \ell^* v_i''(\bar{l}-l_i^*) v_j''(\bar{l}-l_j^*)}{|B|} \left[\tfrac{u_j''(x_j^*)}{v_j''(\bar{l}-l_j^*)} - \tfrac{u_i''(x_i^*)}{v_i''(\bar{l}-l_i^*)} \right] \\ &\times \quad \prod_{k\neq i,j} \left[v_k''(\bar{l} - l_k^*) + w^{*2} u_k''(x_k^*) \right] \end{aligned}$$

Under Assumption 1, we know that $sign \prod_{k\neq i,j} [v_k''(\bar{l} - l_k^*) + w^{*2} u_k''(x_k^*)] = (-1)^{n-2}$ and thus $sign|B| \prod_{k\neq i,j} [v_k''(\bar{l} - l_k^*) + w^{*2} u_k''(x_k^*)] = (-1)^{2(n-1)} > 0$. It follows that $\partial \ell^*(\theta)/\partial \theta_i > 0$ if and only if

$$\tfrac{u_j''(x_j^*)}{v_j''(\bar{l}-l_j^*)} > \tfrac{u_i''(x_i^*)}{v_i''(\bar{l}-l_i^*)}$$

Using (19) and (20), this condition becomes $(\epsilon_l^j(l_j)/\epsilon_x^j(x_j))(l_j/x_j) > (\epsilon_l^i(li)/\epsilon_x^i(x_i))(l_i/x_i)$. But, since consumption and leisure are normal goods, we know from (31) in the proof of Theorem 1 that if $\theta_i > \theta_j$ and $\omega_i = \omega_j$, we get $x_i > x_j$ and $\ell_i < \ell_j$. It follows that $l_j/x_j > l_i/x_i$. The last result is obtained from the fact that $x^*(\theta) = \bar{\omega} + T(\kappa^*, \mu\kappa^*, 1)\ell^*(\theta)$. □

9.7 Proof of Theorem 3

We focus on bilateral transfers between pairs of agents i and j that are elasticity-ordered according to Definition 5. First, consider the case in which

Assumption 3 holds. Let us denote $\zeta_c(\theta) = \rho(T^*\ell^*(\theta)) - \rho_c$ and $\xi_c(\theta) = \gamma(\ell^*(\theta)) - \gamma_c$. Proposition 3 shows that the steady state is saddle-point stable with oscillating convergence if $\rho(x^*) > \rho_c$ and $\gamma(\ell^*) < \gamma_c$. Therefore, an increase of wealth inequality implied by an increase of the share of capital θ_i, and thus a decrease of the share of capital θ_j, leads to damped fluctuations if $\zeta_c(\theta)$ and $\xi_c(\theta)$ are respectively increasing and decreasing with respect to θ_i. We easily get

$$\frac{\partial\zeta_c(\theta)}{\partial\theta_i} = T^*\frac{\partial\ell^*(\theta)}{\partial\theta_i}\left[\rho'(x^*) - \Lambda_c\right], \quad \frac{\partial\xi_c(\theta)}{\partial\theta_i} = \frac{\partial\ell^*(\theta)}{\partial\theta_i}\Big[\gamma'(\ell^*) - \Gamma_c T^*\left[\rho'(x^*) - \Lambda_c\right]\Big]$$

These derivatives are respectively positive and negative in the configurations:

i) if $\partial\ell^*(\theta)/\partial\theta_i < 0$, $\rho'(x^*) < \Lambda_c$ and $\gamma'(\ell^*) > \Gamma_c T^*\left[\rho'(x^*) - \Lambda_c\right]$,

ii) if $\partial\ell^*(\theta)/\partial\theta_i > 0$, $\rho'(x^*) > \Lambda_c$ and $\gamma'(\ell^*) < \Gamma_c T^*\left[\rho'(x^*) - \Lambda_c\right]$.

The final result follows from Lemma 3.

Second, consider the case in which Assumption 4 holds. Let us denote $\zeta_f(\theta) = \rho(T^*\ell^*(\theta)) - \rho_f$ and $\xi_f(\theta) = \gamma(\ell^*(\theta)) - \gamma_f$. Proposition 4 shows that the steady state is locally unstable with oscillating divergence if $\rho(x^*) > \rho_f$ and $\gamma(\ell^*) < \gamma_f$, and γ_f is a flip bifurcation value so that there generically exist period-two cycles, in a left neighborhood of γ_f, which are saddle-point stable. Therefore, an increase of wealth inequality implied by an increase of the share of capital θ_i, and thus a decrease of the share of capital θ_j, leads to persistent fluctuations if $\zeta_f(\theta)$ and $\xi_c(\theta)$ are respectively increasing and decreasing with respect to θ_i. We easily get

$$\frac{\partial\zeta_f(\theta)}{\partial\theta_i} = T^*\frac{\partial\ell^*(\theta)}{\partial\theta_i}\left[\rho'(x^*) - \Lambda_f\right], \quad \frac{\partial\xi_f(\theta)}{\partial\theta_i} = \frac{\partial\ell^*(\theta)}{\partial\theta_i}\Big[\gamma'(\ell^*) - \Gamma_f T^*\left[\rho'(x^*) - \Lambda_f\right]\Big]$$

These derivatives are respectively positive and negative in the configurations:

i) if $\partial\ell^*(\theta)/\partial\theta_i < 0$, $\rho'(x^*) < \Lambda_f$ and $\gamma'(\ell^*) > \Gamma_f T^*\left[\rho'(x^*) - \Lambda_f\right]$,

ii) if $\partial\ell^*(\theta)/\partial\theta_i > 0$, $\rho'(x^*) > \Lambda_f$ and $\gamma'(\ell^*) < \Gamma_f T^*\left[\rho'(x^*) - \Lambda_f\right]$.

The final result follows from Lemma 3. □

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