General equilibrium effects of changes in exogenous government budget deficit restrictions in efficient economies[1]

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Summary. We consider an intertemporal general equilibrium model with production and analyze how restrictions on the government debt path affects the set of equilibrium allocations. The model is fairly general as agents are heterogeneous, there are many traded consumption commodities, many productive inputs and the technology is reasonably general. Importantly, agents face restrictions on the access to the credit market. We assume that the government has access to both lump-sum and consumption taxes but that these are anonymous. We investigate the circumstances for which the irrelevance result a la Riccardo holds. The necessary conditions refer to the extent of the restrictions, the size of the set of available tax instruments and the smoothness of the underlying technology. We show that there are open sets of economies satisfying these assumptions. However, when all the consumption goods are produced and the technology is regular and smooth, GBDR typically matter.

Keywords: Optimal taxation, Balanced-Budget Amendment, Consumption Taxes, Endogenous Credit Constraints, Government Budget Deficit Irrelevance, Lump-sum Taxes, Heterogeneous Consumers. Smooth Production. Non-substitution.

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1 Introduction

There is an abundant literature arguing that the sequence of the government debt is irrelevant[2]. However, these results implicitly assume that consumers face no credit restrictions, so that they can all participate in the effort to borrow on behalf of the government. In presence of borrowing constraints or restricted access to risk-free bond markets irrelevance typically does not hold. The issue is serious as the amount that has to be borrowed may be quite large from the perspective of the consumer. Here we analyze to which extent the presence of agent heterogeneity, distortionary consumption taxes may be used to redistribute the burden of borrowing on "behalf of the government" on the most liquid consumers and restore a weak form of irrelevance.

We adopt an intertemporal deterministic model with heterogeneous agents and production. There are many traded consumption goods, many productive inputs, the technology is reasonably general and there are heterogeneous agents. The set of tax instruments is quite general. We do not exclude lump-sum taxes and transfers as often assumed by the literature. However, we assume that taxes cannot be individual specific although the distribution of the heterogeneous agents is known. We also assume for simplicity that each commodity can be taxed at its own rate. In the model, private credit markets are imperfect in the sense that consumers can only borrow up to the present value of their future endowments in the physical commodities that can be used as collateral. The other assumptions are more or less standard, as the fact that individuals are completely rational in the sense that they use perfect forecast and that the cost of administration of the tax schedule is negligible. Finally, there is a government that produces a public good purchasing the necessary inputs from the producers. However, we assume that the government demand is exogenously fixed.

Reforms in government budget deficit restrictions may affect social welfare, individual utilities and individual allocations. The government budget deficit restriction is said to be *irrelevant* if the set of achievable equilibrium allocations is unaffected by the change in the government budget deficit restriction. Otherwise, the restriction is said to be *relevant*. This notion of irrelevance is very strong as it applies to all government budget deficit restrictions. For example, even though it seems desirable, a completely balanced government budget is in many instances hardly realistic. There is a need to qualify situations in which the government budget deficit can be at least reduced if not entirely removed. As in Ghiglino and Shell (2000), a government budget deficit restriction is said to be *locally irrelevant* if it is irrelevant for restrictions that are "near to" the base-line deficit, i.e., only period-by-period deficits that are not too far from the baseline deficits

[2] The seminal Barro (1974) result on infinite horizon economies with lump-sum taxation has been extended to a variety of situations. For ex.., to economies with overlapping generations and distortionary taxation (see e.g. Kotlikoff (1993), Kelly (1991), and Ghiglino and Shell (2000)). The results have also been extended to taxes depending on previous activity (Bassetto and Kocherlakota (2004) and Kotlikoff (2003)).

are considered. Finally, *welfare irrelevance* in which the focus is on the achievable utilities is also considered.

Beside the issue of the effect of the reform on the set of achievable allocations or welfare, we also investigate how the composition of the optimal tax scheme is affected by the reform. When consumers face no borrowing constraints, anonymous lump-sum taxes are sufficient to neutralize the effect of any change in the government budget deficit restriction. In this case, the composition of the optimal tax is not affected further by the reform. However, one can suspect that the existence of individual borrowing constraints generates an additional scope for redistributive taxes. Under the constraint that all taxes should be anonymous, consumption taxes seem then useful to reach the welfare maximum because of their redistributive power.

The main results of the paper are summarized below.

1. When financial markets are perfect, anonymous lump-sum taxes are sufficient to achieve irrelevance and the maximal attainable welfare is unaffected by the change in the restriction.

2. With imperfect consumer credit markets, welfare irrelevance may not hold if only anonymous lump-sum taxes are to be used. In a pure exchange economy local irrelevance holds in the presence of endogenous credit constraints provided there exist a sufficiently large number of taxable commodities.

3. In productive economies, if technology is perfectly smooth and all consumption goods need to be produced, allocation and welfare irrelevance usually does not hold and a reform in the GBDR is expected to have a real effect on the economy. On the other hand, exact welfare local irrelevance may be achieved if some inputs are non-substitutable or some consumption goods are supplied as endowments, i.e. not produced. In general, low substitution among inputs reduces the effect of the reform on welfare and individual allocation.

Some remarks on these results are in order. First, why local budget deficit irrelevance may require consumption taxes? With only anonymous lump-sum taxation, if taxes must be increased on the young in order to reduce the deficit, constrained consumers will typically have to decrease their early-life consumptions. This means that the deficit restriction is relevant. With anonymous consumptions taxes, the government will be able to accommodate local changes in the deficit restriction if there are sufficiently many types of commodities to tax. This is because, by altering tax rates, the government is able to affect early-life incomes and late-life incomes while generating the necessary revenue.

Second, saying that the restriction is irrelevant is not saying that the restriction does not matter. If the restriction either directly or indirectly affects expectations in such a manner that it affects the selection of the equilibrium, then the restriction does matter.

In case of economies with multiple equilibria, changing the tax scheme typically affects all equilibria. So, the tax may be optimal for one selection but not optimal for a different one. We then assume that the government always pick the best one (see Diamond (1982) and Keister and Hennis (2009)).

A third remark is that in standard smooth general equilibrium most of the results obtained for pure exchange economies can be extended with little difficulty to production economies. As summarized above, in pure exchange economies a sufficiently large number of consumption tax instruments ensures local irrelevance. Introducing production is not innocuous because the prices charged by the producers for their outputs and those they pay for the inputs admit a unique normalization. Then all the relative prices on the revenue side of the individual budget constraint are fixed (because they can be normalized only once). However, the non-substitutability in some inputs produces the kind of kinks in the production possibility frontier that mimics a pure exchange economy. In fact, what matters for irrelevance is the number of non-substitutable inputs and the number of primary consumption goods.

Fourth, it may seem that a sufficiently rich tax and transfers scheme would achieve all goals: finance the production of the public good, relieve the consumers from their credit constraints and make the government budget deficit meet the restriction. This is not entirely true, though. Indeed, even when the number of tax instruments is sufficient the scheme could fail because of the lack of bonafidelity, i.e. money losing its value, or simply because the taxes are too large and lead to negative selling prices.

In the paper we mainly focus on the effect of changes in the GBDR on the set of achievable equilibrium allocations. Provided the government maximizes some sort of social welfare knowing how the set of allocations is affected is all what is needed to know the reaction of the government to a change in the GBDR. A slightly different question concerns the composition of the optimal tax scheme for a given GBDR. The need for redistribution in order to achieve a welfare maximum in a framework with anonymous tax instruments seems to call for consumption taxes. In fact, the standard Ramsey intuition is that the deadweight losses are close to zero for any marginal dollar risen by taxation, so that at the optimum all tax rates are non zero.

In the model we consider only consumption taxes so that production is always efficient. Introducing other types of taxes may destroy production efficiency. The literature has focused on both factor taxation and intermediate goods taxation. These results indicate that inefficiency is likely to arise in the present model when taxes other than on consumption and lump-sum are introduced. We didn't analyze how the results are affected by this change. However, we expect that overall conclusions do not change much.

Consumption taxation in a model with borrowing constraints was analyzed in Ghiglino and Shell (2003). However, the framework and aim of the analysis was completely different. First, the present analysis focuses on the set of allocations that maximize social welfare and considers the effect of the GBDR on welfare and only subsequently on allocations. Second, we introduce production and show that this has a dramatic effects on the results. Third, we refine the modeling of imperfections and let the level of restriction in the private credit markets to be endogenously determined. Indeed, consumers can only borrow up to the present value of their future endowments in the physical commodities that can be used as collateral. Finally, we exclude generational overlap. This prevents the planner to tax at different rates the different cohorts alive in a given period. The irrelevance result obtained is then stronger as all taxes applied in a given period are identical.

In the present analysis we do not let taxes depend on the state of the economy in previous periods, as proposed in Kotlikoff (1993) and Bassetto and Kocherlakota (2004). However, also in this case restriction on the access to the market for risk-free bonds prevents irrelevance to hold. It is possible that the mechanism used in the present paper to reestablish irrelevance when agents face borrowing constraints may be exploited in the framework with "delayed taxes".

The paper has the following structure. In Section 2 we introduce the model while in Section 3 we describe the fiscal policy. In Section 4 we define the equilibrium, in Section 5 we describe the welfare function while in Section 6 we define irrelevance. The analysis of allocation irrelevance results for pure exchange economies is pursued in Section 7. Section 8 deals with the welfare analysis while Section 9 considers the case of taxes on factors of production and intermediate goods leading to production inefficiencies. The conclusion is confined to Section 10.

2 The Model

We employ an intertemporal model with heterogeneous agents extending over T periods. There are l perishable commodities in every period of which l_c are consumption goods. The l commodities are subdivided in l_o primary commodities and l_p produced commodities. It is assumed for simplicity that in a given period a commodity is either produced or is primary. We also assume that no commodity can be consumed and be used as an input in production. By some abuse of notation, we note indifferently R^{l_o} the set of vectors with l_o coordinates or with l_o non-zero coordinates.

2.1 Agents

There are n agents living for T periods. The behavior of agent h ($h = 1, 2, ..., n$) is described by

maximize $u_h(x_h^1, ..., x_h^T, g)$

subject to

$$\begin{aligned} (p^s + \tau^s) \cdot x_h^s + x_h^{s,m} &= p^s \cdot \omega_h^s + m^s + x_h^{s-1,m} + \delta_{t-1} p^1 \cdot \sum_{j \in J} y_j^{2,1} \theta_{i,j} \qquad (1) \\ x_h^{s,m} &\geq -\sum_{t=s+1}^{T} p_C^t \cdot \omega_{Ch}^t, s = 1, ..., T-1 \\ x_h^{0,m} = x_h^{T,m} &= 0 \end{aligned}$$

where $x_h^{sm} \in \mathbb{R}$ is the gross addition to money holding in period s by consumer h (see remark below concerning the price of money). Let the share of agent h in the output of firm j in the initial period 1, $y_j^{2,1}$, be $\theta_{h,j}$, with $\sum_{i \in I} \theta_{i,j} = 1$. The Dirac distribution δ_{t-1} takes the value 0 for all t except when $t = 1$, in which case $\delta_0 = 1$. The utility function has the standard properties. In particular, it is twice differentiable with strictly positive first-order derivatives and with corresponding negative definite Hessian.

The remaining notation is as follows. $m^s \in \mathbb{R}$ is the lump-sum money transfer to a consumer in period s; if m^s is negative, then the consumer is paying a lump-sum tax. Following Ghiglino and Shell (2000), $\tau^{si} \in \mathbb{R}$ is the present tax rate levied on a consumer on his consumption of commodity i in period s. Then $\tau^s = (\tau^{s1}, ..., \tau^{si}, ..., \tau^{sl}) \in R^{\ell_c}$ is the vector of anonymous consumption tax rates in period s for the consumers. We also define $m = (m^1, ..., m^T) \in R^T, \tau = (\tau^1, ..., \tau^T) \in R^{T\ell}$. Let $p^s = (p^{s1}, ..., p^{si}, ..., p^{s\ell}) \in R_{++}^{\ell}$ be the vector of present (before-tax) prices for commodities available in period s. The present after-tax vector of commodity prices facing consumers in period s is $p^s + \tau^s \in R_{++}^{\ell}$. Let $x_h^s = (x_h^{s1}, ..., x_h^{si}, ..., x_h^{sl}) \in \mathbb{R}_{++}^{\ell}$ be the vector of consumption in period s by individual h and $\omega_h^s = (\omega_h^{s1}, ..., \omega_h^{si}, ..., \omega_h^{s\ell}) \in \mathbb{R}_{++}^{\ell}$ be the vector of endowments in period s of individual h for $s = 1, 2, ... , T$ and $h = 1, ..., n$. Finally, define the following quantity sequences: $x_h = (x_h^1, ..., x_h^T) \in R_{++}^{T\ell_c}, \omega_h = (\omega_h^1, ..., \omega_h^T) \in R_{++}^{T\ell_o}, x = ((x_h)_{h=1}^{h=n}),\ \omega = ((\omega_h)_{h=1}^{h=n})$.

Some remarks are in order. First, we assume that the use of capital markets is constrained, viz. some individuals face constraints on their borrowing. In particular, we assume that for these consumers in each period borrowing should not exceed the present value of the future endowments in the commodities that can be used as collateral. Letting some physical goods play the role of collateral is standard. Of course, if all commodities can be used as collateral the consumer is not credit restricted. The borrowing constraint is not binding on consumer h if in equilibrium $x_h^{s,m} > -\sum_{t=s+1}^{T} p_C^t \cdot \omega_{Ch}^t$ for $s = 1, ..., T-1$. When x_h^{sm} is negative consumer h is borrowing in period s. Note that future lump-sum transfers cannot be used as collateral. If this were the case then the borrowing restriction would play no role in the present exercise. Lump-sum taxes and transfers would always suffice to obtain irrelevance of GBDR.

Second, we have implicitly assumed that for at least one consumer none of his borrowing constraints is binding. The usual no-arbitrage argument can then be used to establish that the present price of money is constant, i.e., $p^{s,m} = p^{s+1,m} = p^m \in \mathbb{R}_+$ where $p^{s,m} \in \mathbb{R}_+$ is the present price of money in period $s = 1, 2, \cdots, T$. Assuming that the economy is in proper monetary equilibrium, we can set $p^m = 1$[3]. The nominal (coupon) rate of interest on money is assumed without loss of generality to be zero[4]. Hence the only return on holding money is the capital gain relative to commodities. Condition (2) is thus that money appreciate in value relative to any commodity at the commodity rate of interest.

Third, consumers for which the credit restriction is not binding face

$$\sum_{s=1}^{T}(p^s + \tau^s) \cdot x_h^s = \sum_{s=1}^{T} p^s \cdot \omega_h^s + m^s + p^1 \cdot \sum_{j \in J} y_j^{2,1} \theta_{i,j}$$

for $h = 1, 2, ..., n$. The transfers $m_t = (m^1, ..., m^T) \in \mathbb{R}^T$ affect the behavior of the consumer only through the lifetime transfer $\mu = \sum_{s=1}^{T} m^s \in \mathbb{R}$.

2.2 Firms

We now focus on technology. There are two types of firms. Firm j transforms inputs in period t, $y_j^{1,t} \in R_+^l$, into outputs in period t or $t+1$, $y_j^{2,t} \in R_+^{l_p}$ or $y_j^{2,t+1} \in R_+^{l_p}$ depending on the type of the firm. The firms are called "intertemporal" and "infratemporal". Without loss of generality and to avoid useless complexity we assume that infratemporal firms produce only consumption goods. Intertemporal firms are assumed to produce only non consumable goods that are only used as inputs by other firms, the leading example is the capital good. Let the technology be characterized by production functions, noted F, possessing the usual neoclassical properties: $F_j^1 : R_+^{Tl} \to R_+^{Tl_p}$ and $F_j^2 : R_+^{Tl} \to R_+^{Tl_p}$. The net profits of firm j are as usual given by

$$p^0 \cdot y_j^{2,0} + \sum_{t=1}^{T}(-p^t \cdot y_j^{1,t} + p^t \cdot y_j^{2,t} + p^{t+1} \cdot y_j^{2,t+1})$$

where $((y_j^{1,t}, y_j^{2,t}, y_j^{2,t+1})_{j \in J})_{t=1}^T \in R_+^{Tl} \times R_+^{Tl_p} \times R_+^{Tl_p}$ satisfies one of the two following relationships depending on the type of firm,

$$y_j^{2,t+1} \leq F_j^1(y_j^{1,t}) \text{ and } y_j^{2,t} = 0 \text{ or } y_j^{2,t} \leq F_j^2(y_j^{1,t}) \text{ and } y_j^{2,t+1} = 0 \text{ for all } t \text{ and } j.$$

[3]Strictly speaking, setting $p^m = 1$ is not without loss of generality. We know, however, that we can reconstruct the full set of perfect-foresight equilibria by using the absence-of-money-illusion property.

[4]This is because the super-neutrality of money.

Firms are assumed to maximize profits as defined above. This assumption is fully justified only when all the agents owning shares of the initial capital stock are never constrained in their borrowing. They are various situations in which this happens. One can simply assume that constrained consumers are not allowed to participate in the market for shares at all. More naturally one can assume that the constrained consumers do not wish to own shares at any point of time. Finally, the analysis is valid when the focus is on situations in which none of the agents borrowing constraints bind.

Assuming profit maximization is equivalent to assume that firms maximize their instant profit because technologies display intertemporal separability. For example, firms of the intertemporal type are characterized by the following maximization problem

$$\max \quad -p^t \cdot y_j^{1,t} + p^{t+1} \cdot y_j^{2,t+1}$$

$$\text{s.t.} \quad y_j^{2,t+1} \leq F_j^2(y_j^{1,t})$$

We assume constant returns to scale technologies and no distortionary taxes on inputs or outputs. Therefore, profits are zero at the optimum . Decreasing returns firms could be included at no additional cost provided the government can tax away the profits.

As we focus on constant returns to scale technologies, production sets are typically not strictly convex. This opens the possibility to decompose the economy in separate disconnected sets, allowing for independent price normalization. This kind of separability would facilitate irrelevance. We then make the following assumption to rule out that the economy can be decomposed in blocs.

Assumption. Let I_j be the set of indices of inputs in period t used by a firm to produce an output j in the same period t. Then we assume that for any pair j and j' there exist $j_1, j_2, ..., j_k$ such that $(I_j \cap I_{j_1}) \neq \emptyset, (I_{j_1} \cap I_{j_2}) \neq \emptyset, ..., (I_{j_{k-1}} \cap l_{j'}) \neq \emptyset$. We also assume that there is at least one firm using an input in period t to produce an output in period $t + 1$.

3 Fiscal policy

We assume that the government has at its disposal anonymous lump-sum taxation and anonymous consumption taxation. In other words, we assume that lump-sum taxes and consumption tax rates must be the same for every consumers, but that consumption taxes can vary freely over the l_c consumable commodities. General consumer tax classes and more general commodity tax classes could also be considered (see Ghiglino and Shell (2000)). The government's fiscal policy is the sequence of anonymous lump-sum transfers m and the sequence of consumption tax rates τ. Note that the taxes are on the transactions between the consumption sector and the production sector. Production is assumed to present no distortions, competition ensuring that the economy is on a point on the production possibility frontier. We will relax this assumption in the last section of the paper.

Let d^t be the present commodity value (and also the dollar value) of the government budget deficit incurred in period t. Hence we have for the case of lump-sum taxation

$$d^t = p^t g^t + nm^t$$

for $t = 1, 2, ..., T$ where n is the number of consumers. For the case with consumption taxes

$$d^t = p^t g^t - \sum_{h=1}^{n} \sum_{i=1}^{l} \tau^{ti} x_h^{ti} + nm^t$$

for $t = 1, 2, ..., T$. Let d denote the sequence $(d^1, ..., d^{T-1})$. Let δ^t be the present value (and money value) of the constitutionally imposed deficit restriction in period t. Let δ denote the sequence $(\delta^1, ..., \delta^{T-1})$. The budget deficit restriction is then

$$d \leq \delta.$$

According to the previous definition, the deficit is denominated in Arrow-Debreu units of accounts, i.e. money. However, the results usually do not depend on this convention and still hold for deficits denominated in real terms.

4 Equilibrium

We maintain throughout this paper some strong assumptions. We suppose perfect-foresight on the part of consumers and the government. We also suppose that the government is able to perfectly commit to its announced fiscal policy.

Next we define equilibrium in the economy with taxes.

Definition. A competitive tax equilibrium (x, y, g, m, τ, p). *Given the sequence of endowments in primary commodities ω, the feasible fiscal policy m and τ, the exogenous consumption g, the behavior of consumers and firms described by the systems (1), (2) and (3), the numeraire choice yielding p^{11} =1, the (further) monetary normalization yielding $p^m = 1$, a competitive tax equilibrium is defined by a positive price sequence p a consumption allocation sequence x and a production sequence y such that markets clear, so that we have*

$$g^t + \sum_{h=1}^{h=n} x_h^t = \sum_{h=1}^{h=n} \omega_h^t - \sum_{j=1}^{j=J} y_j^{1,t} + \sum_{j=1}^{j=J} y_j^{2,t}$$

for $t = 1, 2, ..., T$ and where J includes all types of firms.

The set of equilibria is denoted E. One may expect the existence of competitive equilibrium to be guaranteed in "nice" intertemporal economies, but this does not extend to our definition. There are three reasons that competitive equilibrium as defined above could fail to exist. The first reason is because we are seeking a *proper* monetary equilibrium, one for which the price of money is strictly positive. For a proper monetary equilibrium to exist the fiscal policy must be bonafide, i.e. there should be no outside money[5]. It should be noted that in a finite horizon economy, a necessary condition for money to have a strictly positive value is the policy to be balanced, i.e. $\sum_{t=1}^{T} d^t = 0$. Therefore, at an equilibrium $d^T = -\sum_{t=1}^{T-1} d^t$.

The second reason applies only to commodity taxation. It might not be possible to equilibrate supply and demand while maintaining the positivity of the two price sequences p and $p + \tau$. The third reason is that equilibrium may fail to exist because of excessive government consumption.

5 Irrelevance of government budget deficit restrictions

In the next section we focus on the conditions such that the government is able to "obey" the restrictions on its deficit without changing the equilibrium allocation. When this is possible the deficit restriction is said to be *irrelevant*. In section X we will see that in many significant situations this notion of irrelevance is equivalent to the more general situation in which neither the *government consumption* nor the *utility of any private consumer* are affected by the GBDR. The GBDR is then said *utility-irrelevant*. Finally, we will also consider *welfare irrelevance*.

Definition. Irrelevance of the deficit restriction. *Let g be government consumption and let (x, y) be an allocation that can be implemented as a competitive equilibrium with some feasible fiscal policy (m, τ) and with the resulting budget deficits given by the sequence d. The deficit restriction $\delta = d$ is said to be irrelevant at (x, y) if for any other deficit restriction sequence δ' there exists a feasible fiscal policy (m', τ') that implements the allocation x as a competitive equilibrium and is compatible with g, but with the resulting deficit d' satisfying $d' \leq \delta'$.*

The above notion of irrelevance is very strong because it involves any possible government budget deficit sequence other than the pre-reform, or baseline, deficit d. In many situations, this type of irrelevance does not obtain because the new competitive equilibrium does not exist, as explained above. A weaker notion of irrelevance focuses only on restrictions "near to" the base-line deficit, i.e., only period-by-period deficits that are not too

[5]See Balasko and Shell (1986, 1993) and Ghiglino and Shell (2000).

far from the baseline deficits are considered. The intent is to qualify situations in which the government budget deficit can be reduced but possibly not completely avoided.

Definition. Local irrelevance of the deficit restriction *Let* (x, y, g, m, τ, p) *be a competitive equilibrium with government budget deficit* d*. The deficit restriction* $\delta = d$ *is said to be locally irrelevant if there is a non-empty open set* $\mathcal{D}$ *of* δ *such that for all* $\delta' \in \mathcal{D}$ *there is* (m', τ', p') *such that* (x, y, g, m', τ', p') *is a competitive equilibrium with government budget deficit* d' *and* $d'^t \leq \delta'^t$ *for all* $t = 1, ..., T-1$.

According to this definition, local irrelevance ensures that at equilibrium the government budget deficit sequence can be made strictly closer to any other imposed sequence of deficits without changing the equilibrium allocation. The notion is "local" as the deficit restriction may not be completely fulfilled. The typical situation is one in which the government budget deficit can be reduced maintaining the original equilibrium allocation but the government budget cannot be fully balanced.

Remark: In finite horizon economies equilibrium requires the fiscal policy to be balanced. This means that if $(d^1, ..., d^{T-1})$ is given, then a unique value of d^T is compatible with the equilibrium (if the equilibrium is unique). In other words, the fiscal reform focuses on a change in the allowed budget deficit during the first $T-1$ periods. In the final period T all debt is paid back.

6 The effects of GBDR in pure exchange economies

The issue we address in this section is how the equilibrium set of a pure exchange economy is affected by a change in the GBDR. More specifically, we investigate whether the government is able to achieve the same allocation despite a reform in the budget deficit restriction. In absence of credit restrictions, irrelevance can easily be obtained with anonymous lump-sum taxes and transfers. However, restrictions on individual credit imply that consumers are unequally affected by anonymous lump-sum taxes and transfers. Irrelevance is then possible only if the tax scheme takes this heterogeneity implicitly into account. Due to differences in preferences and/or initial endowments, taxes that depend on individual consumptions can help to single out the consumers having access to the largest excess liquidity. We start the analysis with an example.

Example Consider a stationary two period economy ($T = 2$) with two commodities per period ($\ell = 2$), two consumers ($n = 2$) and a government consuming in the first period three units of good 1, $g^1 = (g^{11}, g^{12}) = (3, 0)$. Assume that the first consumer faces a credit restriction while the other has free access to the credit market. The constraint on Consumer 1 is that his borrowing should not exceed the present value of his second period endowment in good 2. Preferences and endowments of consumer h are given by:

$$u_h(x_h^1, x_h^2) = \alpha_h \sum_{k=1}^{2} \alpha_{hk} \log x_h^{1k} + (1-\alpha_h) \sum_{k=1}^{2} \beta_{hk} \log x_h^{2k}$$

with $\alpha_1 = 15/16, \alpha_2 = 1/5, (\alpha_{hk})_{h=1,2,}^{k=1,2} = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}, \quad (\beta_{hk})_{h=1,2}^{k=1,2} = \begin{bmatrix} 1/4 & 3/4 \\ 2/5 & 3/5 \end{bmatrix}$ and

	ω_i^{11}	ω_i^{12}	ω_i^{21}	ω_i^{22}
agent 1	5	1	5	2
agent 2	5	5	5	1

The various situations are described in the table below where b_i is the individual saving and Δ_1 the present value of the collateral available to Consumer 1. The first raw corresponds to the situation prior to the reform. The government runs a deficit in the first period and a surplus in the second period. In the second raw the government pays its consumption with a first period lump-sum tax and has a balanced budget. However, the allocation has changed. In the third raw the allocation prior to the reform is restored while there is no deficit in period 1.

	(x_1^{11}, x_1^{12})	(x_1^{21}, x_1^{22})	(x_2^{11}, x_2^{12})	(x_2^{21}, x_2^{22})	b_1	b_2	Δ_1
$\delta_1 = 3$ $\delta_2 = -3$ no tax	$(4.38, 5.00)$	$(0.84, 0.47)$	$(2.62, 1.00)$	$(9.16, 2.53)$	-9.91	12.91	9.91
$\delta_1 = 0$ $\delta_2 = 0$ lump-sum	$(4.30, 4.96)$	$(1.25, 0.67)$	$(2.70, 1.04)$	$(8.75, 2.33)$	-11.10	11.10	11.10
$\delta_1 = 0$ $\delta_2 = 0$ consum. tax	$(4.38, 5.00)$	$(0.84, 0.47)$	$(2.62, 1.00)$	$(9.16, 2.53)$	-21.65	21.65	21.65

QED

The basic intuition on the mechanism at work in the example can be gained simply by counting equations and unknowns. Because the demand of the unconstrained consumer is homogenous of degree zero in prices, there are $T\ell - 1 = 2 \times 2 - 1 = 3$ equations concerning the after-tax relative consumption prices. Furthermore, there are $n = 2$ budget equations, $(T-1)r = 1$ credit restrictions and $T = 2$ government budget equations. The total number of equations is $T\ell - 1 + n + (T-1)r + T = 8$. On the other hand, there are $T = 2$ possible lump-sum taxes and transfers, so that the number of unknowns (with $p^{11} = 1$) is given by $2T\ell - 1 + T = 9$. Therefore, there are more unknowns than equations. Note that when this is the case a solution may still fail to exist simply because of the non-linearity of the system or/and because some coordinate of the solution in prices is negative. Although this last property would be consistent with the formal model it is inconsistent with free disposal of endowments. In the sequel we show that the system is linear. On the other hand, because the magnitude of the deficit restriction matters to cure the negativity problem we focus only on *local* irrelevance.

The next proposition gives a formal general sufficient condition for local irrelevance. This proposition and those that follow hold only generically -i.e., for an open and dense set of economies. In this way, degenerate cases-principally those in which individual endowments are co-linear-are excluded.

Proposition 1 *Let g be government consumption and let x be an allocation that can be implemented as a competitive equilibrium with some feasible fiscal policy (m, τ) and with the resulting government deficits given by the sequence d. Let $r_t, 0 \leq r_t < n$, be the number of consumers for which the credit constraint is binding in period t. Then, if $n + \sum_{t=1}^{T-1} r_t \leq T\ell$ the deficit restriction $\delta = d$ is generically locally allocation irrelevant.*

Proof: Without lack of generality consider the same stationary economy as in the example but with general preferences and endowments. Let the price before tax of the first good in the first period be taken as the numeraire, $\widehat{p}^{11} = 1$. Let $(x_i^{jk})_{i=1,2}^{j=1,2,k=1,2}$ be the equilibrium allocation, $(\widehat{p}^{sk})^{s=1,2,k=1,2}$ be the equilibrium price vector before the reform. Let $((\tau^{sk})^{s=1,2,k=1,2}, (m^s)_{s=1,2})$ be a tax scheme designed to meet the new requirement. If there is irrelevance, the government budget equations read:

$$-\sum_{k=1}^{2}(x_1^{1k} + x_2^{1k})\tau^{1k} + 2m^1 + g^{11} = d^1.$$

$$-\sum_{k=1}^{2}(x_1^{2k} + x_2^{2k})\tau^{2k} + 2m^2 = d^2 = -d^1$$

At the consumer's level, $(x_i^{jk})_{i=1,2}^{j=1,2,k=1,2}$ is an equilibrium allocation provided both the normalized wealths and prices remain unaffected by the policy. Let $(p^{sk})^{s=1,2,k=1,2}$ be the new equilibrium price vector The following equations reflect this:

$$\begin{aligned}
(p^{12} + \tau^{12})/(1 + \tau^{11}) &= \widehat{p}^{12} \\
(p^{21} + \tau^{21})/(1 + \tau^{11}) &= \widehat{p}^{21} \\
(p^{22} + \tau^{22})/(1 + \tau^{11}) &= \widehat{p}^{22} \\
(p^1 \cdot \omega_h^1 + p^2 \cdot \omega_h^2 + m^1 + m^2)/(1 + \tau^{11}) &= \widehat{p}^1 \cdot \omega_h^1 + \widehat{p}^2 \cdot \omega_h^2 + \widehat{m^2} \quad h = 1, 2
\end{aligned}$$

These equations can be made linear in the unknowns (p, τ, m) by multiplication with $(1+\tau^{11})$. However, irrelevance in the presence of credit restrictions requires that Consumer 1 do not borrow more than the value of his endowments in commodity 2 in period 2. Whether this is possible or not depends on how much the consumer is required to borrow on behalf of the government.

When the borrowing constraint is binding, the difference between the first period expenditure and first period income is equal to the present value of the second period collateral, i.e.

$$((p^1+\tau^1)\cdot x_1^1 - p^1\cdot\omega_1^1 - m^1)/p^{22}\omega_1^{22} \;=\; 1$$

In all cases, a sufficient condition is that the difference between the first period expenditure and first period income divided by the present value of the second period collateral is a constant:

$$((p^1+\tau^1)\cdot x_1^1 - p^1\cdot\omega_1^1 - m^1)/p^{22}\omega_1^{22} \;=\; (\widehat{p}^1\cdot x_1^1 - \widehat{p}^1\cdot\omega_1^1)/\widehat{p}^{22}\omega_1^{22}$$

In this example the total system is composed of 8 linear equations in (p,τ,m); 3 equations concerning the normalized prices, 2 concerning the normalized incomes, the additional credit constraint on Consumer 1, and the government budget deficit equations in period 1 and 2. On the other hand, there are 9 variables; 3 commodity prices, 4 consumption taxes and two lump-sum taxes. Even tough there are more variables then equations a solution may fail to exist because it is not assured that the pre-tax price of a consumption good is positive, i.e. we could have for some s ($s=1,2$) and some k ($k=1$, 2) that $p^{sk}<0$. Consequently, the magnitude of the deficit restriction matter and only local irrelevance is expected to hold. The proof could also fail because some of the relevant matrices of the associated linear system are not of sufficient rank (see Ghiglino and Shell (2000)). However, the previous example shows that it is easy to replicate the analysis in Ghiglino and Shell (2000) and show that the result holds for a generical set of endowments. Consequently we do not include these rank calculations here. The proof can then easily be generalized to any number of commodities to obtain the sufficient condition $2T\ell+T-1-(T\ell-1+n+(T-1)r+T)=T\ell-n-(T-1)r\geq 0$. Q.E.D.

Proposition 1 proposes a sufficient but not necessary condition for local irrelevance. Indeed, there are obvious situations in which consumption taxes are not needed for week irrelevance. However, the following result holds.

Proposition 2 *Assume that at a given equilibrium (x,g,m,τ,p) with government budget deficit d the GBDR is binding in some period and that $n+\sum_{t=1}^{T-1} r_t > n > T\ell$. Then the GBDR $\delta=d$ is locally allocation relevant.*

Note that the analysis shows that there are situations in which the use of consumption taxes allows to implement allocations that would not be feasible with anonymous lump-sum instruments.

7 The effects of GBDR in production economies

From the previous section one can get the impression that a large number of consumption tax instruments is enough to ensure local irrelevance. However, a brief look at the proof shows that the mechanism requires some of the prices involved in the income side of the individual budget constraint to be free. Introducing smooth production eliminates many degrees of freedom associated to the income part of the individual budget constraint as the prices paid by the producers for the inputs are in this case linked. Indeed, profit maximization implies that the marginal productivity are related to the factor prices. Consequently, we expect the irrelevance results to be changed dramatically by the introduction of production. This feature is illustrated by the will explored in the second example.

7.1 An example with non-substitution in production

The assumption of smoothness of the production possibility frontier is crucial for this to occur. Indeed, non-substitutability in inputs produces the kind of kinks that makes the economy similar to pure exchange. In the present example we assume that there are five non-substitutable inputs which are used in conjunction with one substitutable "labor" input by two infratemporal firms. Let commodity 6 be the substitutable input, and commodities 7 and 8 be the produced consumption goods. Assume furthermore that inputs 1 to 5 are non-substitutable in both (types of) firms. The production function for a firm i that produces good $i, i = 7, 8$, using inputs and outputs of the same period can be written as

$$F^i(y_i^t) = F^i(\min[y_i^{t1}, y_i^{t2}/c_i^2, ..., y_i^{t5}/c_i^5], y_i^{t6}) \tag{2}$$

Due to the non-substitutability property in these sectors and independently of the relative prices of these inputs, the actual production plan is such that

$$y_i^{t1} = y_i^{t2}/c_i^2 = ... = y_i^{t5}/c_i^5 \tag{3}$$

The program of the firm is to maximize profits

$$\text{maximize } p^{ti}F^i(y_i) - y_i^{t1}\sum_{k=1}^5 p^{tk}c_i^k - p^{t6}y_i^{t6} \tag{4}$$

It useful to define for $i = 7, 8$, the following reduced production functions

$$G^i(y_i^{t1}, y_i^{t6}) = F^i(y_i^{t1}, y_i^{t2}, ..., y_i^{t5}, y_i^{t6}) \text{ with } y_i^{tk} = c_i^k y_i^{t1} \tag{5}$$

The first order conditions are then

$$\begin{aligned} p^{ti}\tfrac{dG^i}{dy^6} &= p^{t6} \text{ for all }, i \in \{7,8\}, t \in \{1,2\} \\ p^{ti}\tfrac{dG^i}{dy^1} &= \textstyle\sum_{k=1}^5 c_i^k p^{tk} \text{ for all }, i \in \{7,8\}, t \in \{1,2\} \end{aligned} \tag{6}$$

The first four equations imply that for a given consumption plan once p^{17} and p^{27} are chosen p^{18} and p^{28} are determined. Similarly, they imply that p^{16} and p^{26} are determined. The second block of four equations will be considered later.

We assume that there is an intertemporal firm using non-substitutable inputs 4 and 5 from the previous period together with current divisible labor to produce current commodity 5. Let

$$y^{25} = F_2^{55}(y^{14}, y^{15}, y^{26}) = G^{55}(y^{14}, y^{26}) \text{ with } y^{15} = c_4^{55} y^{14}$$

Profit maximization provides a link between the two periods.

$$\begin{aligned} p_2^5 \tfrac{dG^{55}}{dy_1^4} &= p_1^4 + c_2^{55} p^{15} \\ p_2^5 \tfrac{dG^{55}}{dy_2^6} &= p^{26} \end{aligned}$$

The second equation implies that p^{25} is determined because p^{26} is determined. Consequently, the first equation of this block indicates that p^{15} can be considered as a function of p^{14}, i .e. $p^{14}(p^{15})$. Consider now profit maximization in period 1.

$$p^{1i} \tfrac{dG^i}{dy^1} = \textstyle\sum_{k=1}^5 c_i^k p^{1k} \text{ for all } i \in \{7, 8\}$$

As p^{11}, p^{17} and p^{18} are given and $p^{15}(p^{14})$, the two equations leave free only p^{14}. In the second period, the similar set of equations leave on top of p^{24} also p^{21}.

We assume that the government uses a given quantity g^{11} of good 1 in the first period to produce the public good. We also assume that prior to the reform the government runs a deficit in the first period. The debt is paid back in the second period with a lump sum tax $\widehat{m}^2$ which also covers second period consumption. Assume that a restriction on the government budget deficit is put in place: the first period budget is required to be balanced.

There are two consumers. The first consumer faces a credit restriction while the other has free access to the credit market. The constraint on Consumer 1 is that his borrowing should not exceed the present value of his second period endowment in good 2. Let $(x_i^{jk})_{i=1,2}^{j=1,2,k=7,8}$ be the equilibrium allocation, $(\widehat{p}^{sk})_{s=1,2,}^{k=7,8}$ be the equilibrium price vector before the reform. Let $((\tau^{sk})_{s=1,2,}^{k=7,8}, (m^s)_{s=1,2})$ be a tax scheme designed to meet the new requirement. If there is irrelevance, the government budget equations read:

$$-\sum_{k=7}^{8} (x_1^{1k} + x_2^{1k}) \tau^{1k} + 2m^1 + p^{11} g^{11} = 0.$$

$$-\sum_{k=7}^{8} (x_1^{2k} + x_2^{2k}) \tau^{2k} + 2m^2 = 0$$

At the consumer's level, $(x_i^{jk})_{i=1,2}^{k=7,8,j=1,2}$ is an equilibrium allocation provided both the normalized wealths and prices remain unaffected by the policy. Let $(p^{sk})_{s=1,2,}^{k=7,8}$ be the new equilibrium price vector. The following five equations reflect this:

$$\begin{aligned}(p^{18}+\tau^{18})/(p^{17}+\tau^{17}) &= \widehat{p}^{18}/\widehat{p}^{17}\\(p^{27}+\tau^{27})/(p^{17}+\tau^{17}) &= \widehat{p}^{27}/\widehat{p}^{17}\\(p^{28}+\tau^{28})/(p^{17}+\tau^{17}) &= \widehat{p}^{28}/\widehat{p}^{17}\\(p^1\cdot\omega_h^1+p^2\cdot\omega_h^2+m^1+m^2)/(p^{17}+\tau^{17}) &= W_h^n(\widehat{p}/\widehat{p}^{17})\\ &= (\widehat{p}^1\cdot\omega_h^1+\widehat{p}^2\cdot\omega_h^2+\widehat{m}^2)/\widehat{p}^{17}\quad h=1,2\end{aligned}$$

The first three equations involve eight possible unknowns. However, counting equations and unknowns is misleading. From the supply side, once p^{17} is chosen p^{18} is determined. The first equation above then gives τ^{18} as a function of τ^{17}. In short, $\tau^{18}(\tau^{17},p^{17})$. In period 1 the government budget deficit reads

$$-(x_1^{17}+x_2^{17})\tau_1^7-(x_1^{18}+x_2^{18})\tau^{18}(\tau^{17},p^{17})+2m^1+p^{11}g^{11}=0.$$

This gives τ^{17} as a function of p^{17}. In second period, once p^{27} is chosen p^{28} is determined. So, the second and third equations above give $\tau^{27}(p^{27})$ and $\tau^{28}(p^{28}(\tau^{27}))$. Then a right choice of τ^{27} allows the second period government budget deficit equation to be fulfilled

$$-\sum_{k=7}^{8}(x_1^{2k}+x_2^{2k})\tau^{2k}+2m^2=0$$

At this stage, only p^{14},p^{17},p^{21} and p^{24} are still free. The credit constraint of consumer 1

$$((p^1+\tau^1)\cdot x_1^1-p^1\cdot\omega_1^1-m^1)/p^{22}\omega_1^{22} \;=\; 1$$

can be satisfied using p^{14}. Finally, we need to consider the two individual budget constraints. In period 1, p^{17} is still available while in period 2, p^{21} and p^{24} are still available. As we have three degrees of freedom (plus two free lump-sum transfers), the two individual budget constraints may be adjusted. Irrelevance is therefore feasible in this example.

7.2 The general case

Consider economies as formalized in Section 2. In particular, consumption commodities are not used as inputs, input commodities are primary commodities or are produced by intertemporal firms and consumers are concerned by the totality of the consumption goods. The initial endowments may include inputs as well as endowments of the consumption goods: let l_c^e endowments in the producible goods, as the consumption goods, and l_i^e endowments in pure input goods. In economies of pure exchange, when the number of consumers with a binding credit constraints in period t is r_t the condition for irrelevance is $T\ell_c\ -1+n+\sum_{t=1}^{T-1} r_t + T \leq 2Tl_c + T - 1$ (see Proposition 1). In that case, l_c is the number of consumption goods. When there is production the condition is more subtle. We first need the following.

Definition *Let i_k be the number of substitutable inputs used in production of output k. Then define $N^t = \sum_{k=1}^{l_p}(i_k^t + 1)$ and $N^{t,t+1} = \sum_{k=1}^{l_p}(i_k^{t,t+1} + 1)$.*

We then obtain the following.

Proposition 3 *Let g be government consumption and let x be an allocation that can be implemented as a competitive equilibrium with some feasible fiscal policy (m, τ) and with the resulting government deficits given by the sequence d. Let N^t and $N^{t,t+1}$ as defined above and let $r_t, 0 \leq r_t < n$, be the number of consumers for which the credit constraint is binding in period t. Then if $\sum_{t=1}^{T} N^t + \sum_{t=1}^{T-1} N^{t,t+1} + n + \sum_{t=1}^{T-1} r_t \leq T\ell$ the deficit restriction $\delta = d$ is generically locally allocation irrelevant.*

Proof. In a given sector all non-substitutable inputs can be "aggregated" in a single input in the sense that only equations related to the marginal productivity of this composite input need to be considered. This appears as a "1" in the above definition. The total number of equations implied by profit maximization of the infratemporal firms is N^t. Intertemporal firms produce $N^{t,t+1}$ further equations. Altogether, there are $N^{t-1,t} + N^{t,t}$ equations characterizing the supply sector in period t. There are also $n + \sum_{t=1}^{T-1} r_t$ budget and credit restriction equations. The total number of equations is then $T\ell_c\ -1+\sum_{t=1}^{T} N^t + \sum_{t=1}^{T-1} N^{t,t+1} + n + \sum_{t=1}^{T-1} r_t + T - 1$. On the other hand, there are $2Tl_c$ prices and taxes associate to consumption goods, $T(l - l_c)$ prices of the non-consumable goods, T lump-sum taxes and -1 due to the normalization. The total degrees of freedom is $Tl_c + Tl + T - 1$. Provided the relevant matrices are full rank there is local irrelevance whenever $T\ell_c\ -1+\sum_{t=1}^{T} N^t + \sum_{t=1}^{T-1} N^{t,t+1} + n + \sum_{t=1}^{T-1} r_t + T \leq T + T(l + l_c) - 1$ or more simply $\sum_{t=1}^{T} N^t + \sum_{t=1}^{T-1} N^{tt+1} + n + \sum_{t=1}^{T-1} r_t \leq Tl$. Applied to the previous example we have $N^1 = N^2 = 2 + 2 = 4$, $N^{1,2} = 2, n = 2, r = 1, T = 2$ and $l = 8$. So the relation is fulfilled with equality as there are 13 equations left and 16 unknowns. The next proposition, gives a general sufficient condition for local irrelevance. The result is generical in the sense explained in the proof of Proposition 1.
Note that even when there are more variables than equations, a solution may fail to exist because the system is non-linear. However, as in the case of pure exchange the system of equations can be transformed into a linear system. On the other hand, it is not assured that p^{sk} is positive as only $p^{sk} + \tau^{sk}$ is constrained to be positive. The fact that for some s ($s = 1, 2$) and some k ($k = 1, 2$) $p^{sk} < 0$ would be consistent with the formal model, but is inconsistent with free disposal of endowments. Consequently, the magnitude of the deficit restriction matter and only *local* irrelevance is expected to hold.

As in the pure exchange case Proposition 3 states a set of sufficient conditions for local irrelevance. Sufficient conditions for local relevance can also be stated. Indeed, the following result holds.

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Proposition 4 *Assume that at a given equilibrium* (x, y, g, m, τ) *with government budget deficit* d *the GBDR is binding in some period and that there exists* t *such that* $r_t > 0$. *If* $\sum_{t=1}^{T} N^t + \sum_{t=1}^{T-1} N^{t,t+1} + n + \sum_{t=1}^{T-1} r_t > T\ell$ *then the GBDR* $\delta = d$ *is generically allocation relevant.*

A trivial case for irrelevance is one in which the GBDR do not bind in any period. There are also obvious situations in which consumption taxes are not needed to achieve week irrelevance, for example when agents do not face a binding credit constraint. Indeed, in this case a lump-sum tax and transfer scheme is sufficient to satisfy locally the new GBDR.

Remark: In the leading example no consumption commodity is also an input. Importantly, such commodities would appear on both sides of the individual budget constraint. A look at the proof shows that if the input is substitutable and not taxable (in order to preserve production efficiency) irrelevance does not hold. On the other hand if it is a non-substitutable input then irrelevance may be proven in a similar way to Proposition 3.

To conclude, exact local irrelevance requires the presence of kinks in the production possibility frontier. One way to have them is to assume that some inputs are non-substitutable. Another favorable circumstance is pure exchange, in which the "production price" of endowments is completely arbitrary. A similar situation arises when some of the consumption goods are supplied as pure endowments. On a more general perspective, Proposition 3 and 4 indicate that when a sufficiently large number of inputs have at some point a small degree of substitutability, consumption taxes may enable the government to keep the welfare effects of the GBDR reform small.

8 The welfare effects of GBDR

The analysis in the previous section focused on the set of attainable allocations. In this section we focus instead on the set of attainable welfare both individual and social. In the type of efficient economy we study here the set of allocations and the set of individual utilities behave similarly. More interesting is to investigate the effect on the social welfare. In this section we address the following question: what are the effects of GBDR reforms on the social welfare. For free we will get a result on the composition of the optimal tax. Social welfare is expected to depends on the consumption of private goods and the public good. However, as the public good is exogenously provided, it can be excluded from the welfare function without lack of generality. We also exclude consumption externalities, i.e. welfare is individualistic. A simple and perhaps even natural choice is to assume that the social welfare function is the weighted sums of the utility functions of the agents.

Definition *Let* $x = (x_i^{sk})_{i=1,2,...,m}^{s=1,...,T,k=1,...,l}$ *be a non-negative allocation and* λ *be a vector of positive weights such that* $\sum_{i=1}^{m} \lambda_i = 1$*. Then the welfare function is defined as* $W_\lambda(x) = \sum_{i=1}^{m} \lambda_i u_i(x_i)$*.*

Because the set of available tax instruments is restricted we need the following definition.

Definition *Let* $\Gamma_{\tau,m}(\delta)$ *be the set of allocations that can be implemented as a competitive equilibrium with anonymous taxes and transfers and such that the government budget deficit restriction* $d^t \leq \delta^t$ *is satisfied for* $t \in \{1, ..., T-1\}$*. In other words,*

$$\Gamma_{\tau,m}(\delta) = \{x \in R^{Tl_c m}, y \in R_+^{T(l+2l_p)} \mid \exists(m, \tau, p) \in R^T \times R^{Tl} \times R^{Tl}$$
$$\text{such that } (x, y, g, m, \tau, p) \in E,\ d^1 \leq \delta^1, d^2 \leq \delta^2, ..., d^{T-1} \leq \delta^{T-1}\}$$

In this section we assume that the government designs the fiscal policy in order to maximize the social welfare function $W_\lambda(x)$ subject to the implementability constraint, i.e.

$$W_\lambda(\delta) = Max_{\tau,m} \quad W_\lambda(x) \quad s.t. \quad (x, y) \in \Gamma_{\tau,m}(\delta)$$

This also implies that prior to the reform the equilibrium allocation x already maximizes the social welfare function.

Finally we need to restrict the attention to *uniformly more restrictive* GBDR.

Definition *Let be the government budget deficit sequence be* δ*. If the sequence* δ' *satisfies* $\delta'^t \leq \delta^t$ *for all* $t = 1, ..., T-1$ *then the reform is said to be uniformly more restrictive.*

We start the analysis by an example.

Example. Reconsider the example of the previous section but with $\alpha_1 = 1/2$. In the initial situation there is no taxation in the first period while a lump-sum tax is applied in the second period to ensure bonafidelity, i.e. a strictly positive price for money. Consequently, the government finances the production of the public good by running a deficit in the first period. From the actual calculations it appears that no individual credit restriction is binding and the equilibrium is Pareto Optimal. This allocation maximizes the sum of the utility of the two agents weighted by the welfare weights associated to the given Pareto Optimum. Suppose that a new regulation requires a balanced first period government budget. What is the impact on the maximal achievable social welfare? In the first scenario the government proceeds to a lump-sum tax in period 1 in order to meet the requirement. As a result, Consumer 1 credit restriction binds, the equilibrium allocation is modified and social welfare is reduced. In the second scenario a sufficiently rich consumption tax is applied that allows for allocation, and therefore welfare, irrelevance. The values are reported in the table below where u_1 and u_2 are the utilities while W is the social welfare. Note that the credit restriction is not binding as $\Delta_1 > -b_1$.

	x_1^{11} x_1^{12}	x_1^{21} x_1^{22}	x_2^{11} x_2^{12}	x_2^{21} x_2^{22}	b_1	b_2	Δ_1	u_1	u_2	W
$\delta_1 = 3$ $\delta_2 = -3$ no taxes in 1	3.84 4.71	2.75 1.29	3.16 1.29	7.25 1.71	-7.91	10.91	17.79	0.97	1.03	1.00
$\delta_1 = 0$ $\delta_2 = 0$ lump-sum in 1	4.30 4.96	1.25 0.67	2.70 1.04	8.75 2.33	-11.10	11.10	11.10	0.66	1.20	0.93
$\delta_1 = 0$ $\delta_2 = 0$ consum. taxes	3.84 4.71	2.75 1.29	3.16 1.29	7.25 1.71	-92.92	92.92	209	0.97	1.03	1.00

QED
The previous example is peculiar in the sense that prior to the reform the implemented allocation both maximizes the welfare function and is Pareto-optimal. This is a coincidence and is not important for our discussion. Indeed, in the absence of distortionary taxes, any allocation such that both the individual credit constraints and the government budget deficit restriction are not binding is Pareto-Optimal. However, even in this case there is little chance that this allocation maximizes the chosen social welfare function.

The general result concerning the effects on social welfare of a reform in GBDR is a direct consequence of Proposition 3. Indeed, Proposition 3 gives sufficient conditions for allocation irrelevance which can be translated into the following sufficient conditions guaranteeing that a reform in the government budget deficit restriction has no effect on the maximal welfare obtainable with anonymous taxes.

Proposition 5 *Let g be government consumption and let x be an allocation that can be implemented as a competitive equilibrium with some feasible fiscal policy (m, τ) and with the resulting government deficits given by the sequence d. Let N^t and $N^{t,t+1}$ be as defined above and let $r_t, 0 \leq r_t < n$, be the number of consumers for which the credit constraint is binding in period t. Then if $\sum_{t=1}^{T} N^t + \sum_{t=1}^{T-1} N^{t,t+1} + n + \sum_{t=1}^{T-1} r_t \leq T\ell$ the deficit restriction $\delta = d$ is generically locally welfare irrelevant.*

Proof: Let (x, y, g, m, τ, p) be an equilibrium with a government budget deficit sequence $d = \delta$. As the sequence δ' satisfies $\delta'^t \leq \delta^t$ for all $t = 1, ..., T-1$ then $W_\lambda(\delta') \leq W_\lambda(\delta)$. Indeed, assume instead that $W_\lambda(\delta) < W_\lambda(\delta')$. In this case it exists (x, y) in $\Gamma_{\tau,m}(\delta')$ such that $W_\lambda(x) > W_\lambda(\widehat{x})$ for all $(\widehat{x}, \widehat{y})$ in $\Gamma_{\tau,m}(\delta)$. This can be true only if $\Gamma_{\tau,m}(\delta') \subsetneq \Gamma_{\tau,m}(\delta)$. As from the definition is obvious that $\Gamma_{\tau,m}(\delta') \subseteq \Gamma_{\tau,m}(\delta)$ the result follows. Consequently, the optimal reaction of the planner is to modify the tax scheme as to guarantee the same social welfare as prior to the reform, whenever this is possible. Note that as the set of available instruments has not changed, welfare irrelevance is generically equivalent to allocation and utility irrelevance. Allocation irrelevance gives sufficient but not necessary condition. Indeed, it is easy to find economies for which welfare irrelevance is obtained under weaker conditions as for example when the initial GBDR does not bind. The conditions for GBDR welfare relevance can be given in the case considered in Proposition 4.

Proposition 6 *Assume that at a given equilibrium* (x, y, g, m, τ) *with government budget deficit* d *the GBDR is binding in some period and there exists* t *such that* $r_t > 0$. *Then if* $\sum_{t=1}^{T} N^t + \sum_{t=1}^{T-1} N^{t,t+1} + n + \sum_{t=1}^{T-1} r_t > T\ell$ *the GBDR* $\delta = d$ *is locally welfare irrelevant*[6].

Proposition 6 may be considered stronger then Proposition 4 because there might be cases in which allocation irrelevance is not required for welfare irrelevance. Achieving welfare irrelevance may be desirable but is clearly often unfeasible. For example, the number of tax instruments could be insufficient, as in Proposition 6, or inputs could be substitutable although scarcely so. When a sufficient number of inputs is only slightly substitutable the gain from reducing the inefficiency due to the constraints is expected to overcome the cost associated to the distortion induced by the consumption taxes. In this case there is relevance but still the consumption taxes are used

Corollary 1 *There exist an open set of economies such that the optimal tax scheme includes consumption taxes.*

The previous analysis focused the possibility to keep the same social welfare in spite of the reform on the GBDR. This leaves open several issues. First, are there relevant reforms such that welfare irrelevance is not desirable? This is an open question. Another issue is how production inefficiency would affect the results.

9 Conclusion

In the present paper we focus on how a benevolent government would react to a change in the sequence of annual budget deficit restrictions. When financial markets are perfect, anonymous lump-sum taxes are sufficient to achieve irrelevance and the maximal attainable welfare is unaffected by the change in the restriction. With imperfect consumer credit markets welfare irrelevance may not hold. In a pure exchange economy we show that irrelevance still holds in the presence of endogenous credit constraints provided there exists a sufficiently large number of anonymous consumption taxes. In productive economies, the conditions for welfare irrelevance are much more difficult to obtain. If production is perfectly smooth, allocation and welfare irrelevance usually does not hold, and a reform in the GBDR is expected to have a real effect on the economy. On the other hand, exact welfare irrelevance may be achieved if some inputs are non-substitutable or

[6] We assume the government selects its monetary transfers and taxes in order to maximize the social welfare function, but some monetary transfers may be compatible with several equilibria. In the present paper it is simply assumed that the government selects the most favorable equilibrium in case of several equilibria (see Ennis and Keister (2009))

some consumption goods are supplied as endowments. In general, when inputs have a low degree of substitutability the effect of a GBDR reform is expected to have little effect on the achievable welfare maximum. Finally, even when all inputs are substitutable the optimal reaction of the government is expected to include consumption taxes. This may be a factor inducing governments to refrain from using exclusively lump-sum taxes.

The frameworks can be extended in several ways. Besides considering more general tax schemes, as briefly explored in the previous section, an infinite horizon version of model could be constructed. A more sophisticated demographic structure allowing for overlapping generations could also be analyzed. Finally, the supply side of the economy could be formalized in a more general way and government consumption could be endogeneised. We leave these generalizations for future research but we think that they would not change much the message of the present paper.

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