Dynamics in OG economies

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Abstract

In the present paper we consider a class of dynamic economic models in which the equilibrium price for the ℓ goods, $p_t \in \mathbb{R}^{\ell}_{++}$, is obtained as the solution to an autonomous system of the form

$$z(p_{t-S}, \ldots, p_t, \ldots, p_{t+S}) \ = \ 0, \forall t \geq S$$

where $S+1$ is the lifespan of the consumers and $z : \mathbb{R}^{(2S+1)\ell} \rightarrow X \subseteq \mathbb{R}^{(2S+1)\ell}$ is obtained by maximizing individual felicity. We investigate both the existence and number of stationary solutions as well as the existence of periodical solutions.

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1. Introduction

Economies are composed of agents, such as consumers and firms, imbedded in a market structure. Economic models assume that these agents act according to specified behavioral rules, as for example that consumers are utility maximizers. However, the action chosen by each agent depends on the other's actions. The most attractive situation is the one of perfect competition in which the relevant information the agents need in order to take their decisions is conveyed by prices. An economic equilibrium is a situation in which prices are such that all markets clear, i.e. the supply of goods (and factors) is exactly balanced by the demand for these goods. Assuming that the economy expands over a discrete number of periods, an equilibrium price is then formally defined as the solution to a market clearing equation

$$Z(p) \;=\; 0 \tag{1}$$

where $p \in \mathbb{R}_+^{T\ell}$ with T being the horizon of the economy, ℓ the number of available commodities in every period and Z a map from $\mathbb{R}^{T\ell}$ to $\mathbb{R}^{T\ell}$.

A striking fact of real world economies is that consumers and workers eventually die while the rest of the system can be assumed as eternal. The Overlapping Generations Model (Samuelson (1958)) captures this structure. In such models, the equilibrium price is an infinite sequence. However, due to the finite lifespan of the agents, individual excess demand functions involves only a finite number of periods. This imply that equation (1) can be decomposed in finite dimensional blocs. The equilibrium price is then obtained as a solution to a higher order non-linear difference equation of the

$$z_t(p_{t-S}, \ldots, p_t, \ldots, p_{t+S}) \;=\; 0$$

where $S + 1$ is the lifespan of consumers. A reasonable assumption is that the model itself is stationary or time invariant, so $z_t = z$. In this case, stationary solutions as well as periodical solutions may be defined. In fact both have been shown to exist in a variety of versions of the model. Note that cyclical solutions are interesting since they may be interpreted as business cycles.

In principle, the dynamics depends on all the properties of the fundamentals, such as endowments and preferences of consumers and technologies used by firms.

In our model, production is left out so the model is a pure exchange economy. There is no reason to assume that the dynamic properties of the model are independent, even qualitatively, of the characteristics of the consumers. Since

Balasko (1988) it is customary to assume that preferences are fixed and endowments are parameters. This assumption appears to be justified by the fact that an important task for economic theory is to design useful policies and policies are expected to have no influence on the formation of preferences while they typically are designed to affect endowments. In this setup, economies are parameterized by a vector of initial endowments ω and equilibria are described by a correspondence $p(\omega)$.

In the present paper we analyze an overlapping-generations model in which consumers live for several periods. This will make the analysis more complex that the usual in which a lifetime of two is assumed (as in Ghiglino and Tvede (1995)). We will investigate both the existence and number of stationary solutions as well as the existence of periodical solutions. The analysis is carried out for given but generic preferences while endowments are varied. The result will be that for generical preferences there exist endowments such that the economy has cycles and at least $2\ell - 1$ stationary solutions where ℓ is the number of goods available in each period.

The paper is organized as follows: In section 2 the model is introduced while the relevant dynamical system and the solution concepts are defined in Section 3. Section 4 focuses on the existence and number of stationary solutions. In section 5 the stability properties of the steady states are analyzed. Finally, the existence of periodical solutions is investigated in Section 6.

2 The model

A stationary double-ended pure exchange overlapping generations economy (or OG economy for short) is considered so time extends from $-\infty$ to ∞. At every date, a finite number ℓ of commodities are available and a finite number m, where $m \geq 2\ell + 1$, of consumers are born. Consumers live for $S + 1$ dates, where $S \geq 1$, so at every date $(S + 1)m$ consumers are alive.

A consumer is described by a consumption set $X = \mathbb{R}^{(S+1)\ell}$, an endowment vector $\omega_i = (\omega_i^0, \ldots, \omega_i^S) \in X$ and an utility function $u_i : X \to \mathbb{R}$. An OG economy $\mathcal{E}_{OG}$ is described by the consumers in one generation because all generations are identical except for their dates of births, so

$$\mathcal{E}_{\mathrm{OG}} = (\omega_i, u_i).$$

Consumers are supposed to satisfy the following assumptions:

(A.1) $u_i \in C^\infty(X, \mathbb{R})$ and $u_i^{-1}(a)$ is bounded from below for all $a \in \mathbb{R}$;

(A.2) $Du_i(x) \in \mathbb{R}^{(S+1)\ell}_{++}$ for all $x \in X$, and;

(A.3) $y' D^2 u_i(x) y < 0$ for all $x \in X$ and $y \in \mathbb{R}^{(S+1)\ell} \setminus \{0\}$.

The set of functions which satisfy (A.1)-(A.3) is endowed with the Whitney topology and the set of economies is endowed with the product topology.

The problem of a consumer in generation t in an OG economy is to maximize the utility subject to the budget constraint:

$$\begin{aligned} \max \quad & u_i(x^0, \ldots, x^S) \\ \text{s.t.} \quad \sum_s p_{t+s} \cdot x^s \; &= \; \sum_s p_{t+s} \cdot \omega_i^s. \end{aligned}$$

For all prices $(p_t, \ldots, p_{t+S}) \in \mathbb{R}^{(S+1)\ell}_{++}$ and endowment vectors $\omega_i \in X$ there exists a unique solution to the problem. The demand function

$$f_i = (f_i^0, \ldots, f_i^S) : \mathbb{R}^{(S+1)\ell} \times \mathbb{R} \to X$$

associates prices and incomes with solutions. The demand function satisfies:

- *Walras' law*, so

$$\sum_s p_{t+s} \cdot f_i(p_t, \ldots, p_{t+S}, \textstyle\sum_j p_{t+j} \cdot \omega_i^j) \; = \; \displaystyle\sum_s p_{t+s} \cdot \omega_i^s;$$

- *homogeneity of degree zero in prices*, so for all $\alpha > 0$

$$f_i(\alpha p_t, \ldots, \alpha p_{t+S}, \textstyle\sum_s \alpha p_{t+s} \cdot \omega_i^s) \; = \; f_i(p_t, \ldots, p_{t+S}, \textstyle\sum_s p_{t+s} \cdot \omega_i^s);$$

- *boundedness from below*, so for all $\omega_i \in X$ there exists $v \in \mathbb{R}^{(S+1)\ell}$ such that for all $(p_t, \ldots, p_{t+S}) \in \mathbb{R}^{(S+1)\ell}_{++}$

$$f_i(p_t, \ldots, p_{t+S}, \textstyle\sum_s p_{t+s} \cdot \omega_i^s) \; \geq \; v,$$

  and;

- *smoothness*, so $f_i \in C^\infty(\mathbb{R}^{(S+1)\ell}_{++} \times \mathbb{R}, X)$.

3 Equilibria, steady-states and fibers

Definition 1. *An* ***equilibrium*** *is a price system and a collection of individual endowment vectors $((p_t), (\omega_i))$ such that all markets clear, so for all t:*

$$\sum_s \sum_i f_i^s(p_{t-s}, \ldots, p_{t-s+S}, \textstyle\sum_j p_{t-s+j} \cdot \omega_i^j) \; = \; \displaystyle\sum_s \sum_i \omega_i^s.$$

For an OG economy $\mathcal{E}_{\text{OG}} = (\omega_i, u_i)$ equilibria are solutions to an autonomous difference equation in the sense that $((p_t), (\omega_i))$ is an equilibrium if and only if (p_t) is a solution to the autonomous difference equation

$$\sum_s \sum_i f_i^s(p_{t-s}, \ldots, p_{t-s+S}, \textstyle\sum_j p_{t-s+j} \cdot \omega_i^j) \;=\; \displaystyle\sum_s \sum_i \omega_i^s.$$

Following Samuelson (1958), a golden rule steady-state is an equilibrium for which the price vector is constant across dates.

Definition 2. *A* ***golden rule steady-state*** *is an equilibrium* $((p_t), (\omega_i))$ *for which there exists a price vector* $p \in \mathbf{R}^{\ell}_{++}$ *such that* $p_t = p$.

A golden rule steady-state is written $(p, (\omega_i))$ for simplicity.

In the analysis, the collections of individual utility functions are fixed and the relation between the collections of individual endowment vectors and the set of equilibria is in focus. Therefore for every price vector $p \in \mathbb{R}^{\ell}_{++}$ and distribution of incomes (w_i), $w_i \in \mathbb{R}$, the set of collections of individual endowment vectors (ω_i), $\omega_i \in X$, such that $(p, (\omega_i))$ is a golden rule steady state, with $p \cdot \sum_s \omega_i^s = w_i$, is considered. The sets are linear manifolds and are denoted *fibers*.

In Balasko (1989) it is shown that for Arrow-Debreu economies, which are defined in Section 4, the set of Walras equilibria has a fiber bundle structure. As shown in Ghiglino & Tvede (1995) and Balasko & Lang (1998), a similar result applies to OG economies.

Definition 3. *A* ***fiber*** $F(p, (w_i)) \subset \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{(S+1)\ell m}_{++}$*, where* $w_i \in \mathbb{R}$*, is the set of golden rule steady states* $(p, (\omega_i))$ *for which*

(F.1) $p \cdot \sum_s \omega_i^s = w_i$*, and;*

(F.2) $\sum_s \sum_i \omega_i^s = \sum_s \sum_i f_i^s(p, \ldots, p, w_i)$*,*

and a ***restricted fiber*** $F(p, (w_i), (r^s)) \subset F(p, (w_i))$*, where* $r^s \in \mathbb{R}^{\ell}$ *such that* $\sum_s r^s = 0$*, is the set of golden rule steady states* $(p, (\omega_i))$ *in the fiber* $F(p, (w_i))$ *for which*

(F.3) $\sum_i (f_i^s(p, \ldots, p, w_i) - \omega_i^s) = r^s$.

Remark: At golden rule steady states in a fiber, the intensity of trade between generations vary, while at golden rule steady states in a restricted fiber the intensity of trade between generations is constant, because

$$(\sum_i (f_i^0(p, \ldots, p, w_i) - \omega_i^0), \ldots, \sum_i (f_i^S(p, \ldots, p, w_i) - \omega_i^S)) \;=\; (r^0, \ldots, r^S).$$

It is shown in Ghiglino & Tvede (1995) that restricted fibers are non-empty, linear manifolds of dimension $((S+1)\ell - 1)(m-1)$. *End of remark*

All OG economies $\mathcal{E}_{\mathrm{OG}} = (\omega_i, u_i)$, where $(\omega_i) \in \mathrm{pr}_\omega F(p, (w_i))$, have a steady state where the price vector is p and the collection of individual consumption bundles is $(f_i(p, \ldots, p, w_i))$. However, for an OG economy, the collection of individual endowment vectors may be in the intersection of more than one projection of a restricted fiber on collections of individual endowment vectors, so an OG economy may have more than one steady state.

In the next section, properties of the set of golden rule steady states are explored.

4 On the number of steady states

A pure exchange Arrow-Debreu economy (or an AD economy for short) with a finite number of commodities ℓ and a finite number of consumers m is considered. A consumer is described by a consumption set $Y = \mathbb{R}^\ell$, an endowment vector $\pi_i \in Y$ and an utility function $v_i : Y \to R$. An economy $\mathcal{E}_{\mathrm{AD}}$ is described by the consumers, so $\mathcal{E}_{\mathrm{AD}} = (\pi_i, v_i)$.

The problem of a consumer in an Arrow-Debreu economy is to maximize the utility subject to the budget constraint:

$$\begin{array}{rl} \max & v_i(y) \\ \text{s.t.} & p \cdot y \;=\; p \cdot \pi_i. \end{array}$$

If the utility function satisfies (A.1)-(A.3) for $S = 0$, then for all price vectors $p \in R^\ell_{++}$ and endowment vectors $\pi_i \in Y$ there exists a unique solution to the problem. The demand function $g_i : \mathbb{R}^\ell_{++} \times \mathbb{R} \to Y$ associates prices and incomes with solutions to the problem.

Definition 4. *A* ***Walras equilibrium*** *is a price vector and a collection of individual endowment vectors* $((p_t), (\pi_i))$ *such that all markets clear, so:*

$$\sum_i g_i(p, p \cdot \pi_i) \;=\; \sum_i \pi_i.$$

OG economies induce AD economies. Indeed, for an OG economy $\mathcal{E}_{\mathrm{OG}} = (\omega_i, u_i)$, let $\pi_i = \sum_s \omega_i^s$ and $v_i(y) = \max\{u_i(x^0, \ldots, x^S) | \sum_s x^s = y\}$, then $\mathcal{E}_{\mathrm{AD}} = (\pi_i, v_i)$ is an AD economy with ℓ commodities and m consumers.

The following two lemmas explore the relation between OG economies and AD economies.

Lemma 1. *$(p,(\omega_i))$ is a golden rule steady state for the OG economy $\mathcal{E}_{OG} = (u_i, \omega_i)$ if and only if $(p,(\pi_i))$ is a Walras equilibrium for the AD economy $\mathcal{E}_{AD} = (\pi_i, v_i)$.*

Proof: Suppose that $(p,(\omega_i))$ is a golden rule steady state for the OG economy $\mathcal{E}_{\mathrm{OG}} = (u_i, \omega_i)$ and that $(p,(\pi_i))$ is a Walras equilibrium for the AD economy $\mathcal{E}_{\mathrm{AD}} = (\pi_i, v_i)$. Then $f_i(p,\ldots,p,\sum_s p\cdot\omega_i^s)$ solves the problem of consumer i in the OG economy and markets clear

$$\sum_s\sum_i f_i(p,\ldots,p,\textstyle\sum_s p\cdot\omega_i^s) \;=\; \displaystyle\sum_s\sum_i \omega_i^s,$$

if and only if $g_i(p,\pi_i) = \sum_s f_i^s(p,\ldots,p,\sum_s p\cdot\omega_i^s)$ solves the problem of consumer i in the associated AD economy and markets clear

$$\sum_i g_i(p,\pi_i) \;=\; \sum_s\sum_i f_i^s(p,\ldots,p,\textstyle\sum_j p\cdot\omega_i^j) \;=\; \displaystyle\sum_s\sum_i \omega_i^s \;=\; \sum_i \pi_i.$$

Hence, $(p,(\omega_i))$ is a golden rule steady state for the OG economy $\mathcal{E}_{\mathrm{OG}} = (u_i,\omega_i)$ if and only if $(p,(\pi_i))$ is a Walras equilibrium for the AD economy $\mathcal{E}_{\mathrm{AD}} = (\pi_i, v_i)$.
Q.E.D

Lemma 2. *If $u_i : X \to \mathbb{R}$ satisfies (A.1)-(A.3), then $v_i : Y \to \mathbb{R}$ satisfies (A.1)-(A.3) for $S = 0$.*

Proof: Suppose that $u_i : X \to \mathbb{R}$ satisfies (A.1)-(A.3) and for $y \in Y$ consider the following problem

$$\begin{aligned} \max \quad & u_i(x) \\ \text{s.t.} \quad & \sum_s x^s \;=\; y \end{aligned}$$

then x is a solution if and only if there exists $p \in \mathbb{R}^\ell$ such that

$$\begin{aligned} D_0 u(x) - p &= 0 \\ &\vdots \\ D_S u(x) - p &= 0 \\ \sum_s x^s &= y. \end{aligned}$$

For all $y \in Y$ there exists one and only one solution, therefore let $h_i : Y \to X$ be the function which maps the parameter to the solution. Then $v_i(y) = u_i(h_i(y))$

and according to the Implicit Function Theorem

$$\begin{aligned} Dv_i(y) &= \frac{1}{S+1}\sum_s D_s u_i(h_i(y)) \\ D^2 v_i(y) &= \frac{1}{S+1}\sum_j \sum_k D^2_{jk} u_i(h_i(y)). \end{aligned}$$

Hence, if $u_i : X \to \mathbb{R}$ satisfies (A.1)-(A.3), then $v_i : Y \to \mathbb{R}$ satisfies (A.1)-(A.3) for $S = 0$.

Q.E.D

The following two theorems are on existence and multiplicity of golden rule steady states.

Theorem 1. *For all OG economies $\mathcal{E}_{OG} = (\omega_i, u_i)$ there exists a golden rule steady state $(p, (\omega_i))$.*

Proof: Follows directly from Proposition 3.2 in Kehoe & Levine (1984) or according to Lemma 2 from Corollary 4.6.3 in Balasko (1988).

Q.E.D

Theorem 2. *For all $(p, (w_i), (r^s))$ and an open and dense set of collections of individual utility functions (u_i) there exists an open subset of the restricted fiber $F(p, (w_i), (r^s))$ such that if $(p, (\omega_i))$ is in the open subset, then the OG economy $\mathcal{E}_{OG} = (\omega_i, u_i)$ has at least $2\ell - 1$ golden rule steady states.*

Proof: Follows directly from Theorem 1 in Ghiglino & Tvede (1997).

Q.E.D

In the next section, properties of the local dynamics at golden rule steady states are explored.

5 On the local dynamics at steady states

For an OG economy $\mathcal{E}_{\mathrm{OG}} = (\omega_i, u_i)$ equilibria are solutions to a difference equation in the sense that $((p_t), (\omega_i))$ is an equilibrium if and only if (p_t) is a solution to the difference equation

$$\sum_s \sum_i f_i^s(p_{t-s}, \dots, p_{t-s+S}, \textstyle\sum_j p_{t-s+j} \cdot \omega_i^j) \;=\; \displaystyle\sum_s \sum_i \omega_i^s.$$

At a golden rule steady state $(p, (\omega_i))$, let the $\ell \times \ell$-matrices (J_{jk}), where $j, k \in \{0, \dots, S\}$, be defined by

$$J_{jk} \;=\; \sum_i (D_{p_k} f_i^j(p, \dots, p, p \cdot \textstyle\sum_s \omega_i^s) + D_{w_i} f_i^j(p, \dots, p, p \cdot \textstyle\sum_s \omega_i^s) \omega_i^{k\prime}).$$

Both the $\ell \times \ell$-matrix

$$D_{p_k} f_i^j(p, \dots, p, p \cdot \textstyle\sum_s \omega_i^s)$$

and the ℓ-vector

$$D_{w_i} f_i^j(p, \dots, p, p \cdot \textstyle\sum_s \omega_i^s)$$

are constant in the fiber $F(p, (w_i))$ because neither the price vector p nor the income $w_i = p \cdot \sum_s \omega_i^s$ vary. Therefore the $\ell \times \ell$-matrix J_{jk} is an affine function of the endowment vectors $(\omega_1^k, \dots, \omega_m^k)$ in a fiber $F(p, (w_i))$.

Let $G(p, (w_i)) \subset F(p, (w_i))$ be the set of golden rule steady states for which the matrix J_{0S} has full rank. Then, according to the Implicit Function Theorem, for all $(p, (\omega_i)) \in G(p, (w_i))$, there exist neighborhoods $U, V \subset \mathbb{R}^{2S\ell}$ of $(p, \dots, p) \in \mathbb{R}^{2S\ell}$ and $\Omega \subset X^m$ of (ω_i) and a smooth function $\Gamma : U \times \Omega \to V$ such that $(p_{t-S}, \dots, p_{t+S})$, where $q_t = (p_{t-S}, \dots, p_{t+S-1}) \in U$ and $q_{t+1} = (p_{t-S+1}, \dots, p_{t+S}) \in V$, satisfies

$$\sum_s \sum_i f_i^s(p_{t-s}, \dots, p_{t-s+S}, \textstyle\sum_j p_{t-s+j} \cdot \omega_i^j) \;=\; \displaystyle\sum_s \sum_i \omega_i^s.$$

if and only if

$$q_{t+1} \;=\; \Gamma(q_t, (\omega_i)).$$

Furthermore, if $q = (p, \dots, p) \in \mathbb{R}_{++}^{2S\ell}$ and

$$K_s \;=\; \sum_{-j+k=s} J_{jk},$$

then

$$D_q \Gamma(q, (\omega_i)) \;=\; \begin{pmatrix} & I & & \\ & & \ddots & \\ & & & I \\ -K_S^{-1} K_{-S} & -K_S^{-1} K_{-S+1} & \cdots & -K_S^{-1} K_{S-1} \end{pmatrix}.$$

The following lemma characterizes the pairs of eigenvalues and eigenvectors of $D_q \Gamma(q, (\omega_i)$.

Lemma 3. *At a golden rule steady state $(p, (\omega_i)) \in G(p, (w_i))$, $(\lambda, V) \in \mathbb{C} \times \mathbb{C}^{2S\ell}$ is a pair of an eigenvalue and an eigenvector of $D_q\Gamma(q, (\omega_i))$ if and only if*

$$V = \begin{pmatrix} v \\ \lambda v \\ \vdots \\ \lambda^{2S-1} v \end{pmatrix},$$

and

$$\sum_s \lambda^{S+s} K_s v = 0.$$

Proof: Follows directly from the fact that at a golden rule steady state $(p, (\omega_i)) \in G(p, (w_i))$, $(\lambda, V) \in \mathbb{C} \times \mathbb{C}^{2S\ell}$ is a pair of an eigenvalue and an eigenvector of $D_q\Gamma(q, (\omega_i))$ if and only if

$$D_q\Gamma(q, (\omega_i))V - \lambda V = 0.$$

Q.E.D

The following definition and theorem characterize the pairs of eigenvalues and eigenvectors of $D_q\Gamma$ on restricted fibers. For a complex vector $V \in \mathbb{C}^k$, let re$V$ be the real part of V, im$V$ be the imaginary part of V and $\bar{V}$ be the conjugate vector of V.

Definition 5. *$\mathcal{S}(p)$ is the set of collections of pairs of eigenvalues and eigenvector $((\lambda_1, V_1), \ldots, (\lambda_{2S\ell}, V_{2S\ell}))$, where $(\lambda_s, V_s) \in \mathbb{C} \times \mathbb{C}^{2S\ell}$, such that:*

(B.1) for all s there exists $v_s \in \mathbb{C}^\ell$ such that

$$V_s = \begin{pmatrix} v_s \\ \lambda_s v_s \\ \vdots \\ \lambda_s^{2S-1} v_s \end{pmatrix};$$

(B.2) $\lambda_{2S\ell} = 1$ and $v_{2S\ell} = p$, and;

(B.3) for all subsets a of $\{1, \ldots, 2S\ell - 1\}$, such that $|a| \leq (S+1)\ell - 1$ and if $j \in a$ then $k \in a$ for $(\lambda_k, V_k) = (\bar{\lambda}_j, \bar{V}_j)$, the $((S+1)\ell) \times (|A|+1)$-matrix

$$\left(\left(re \begin{pmatrix} v_s \\ \vdots \\ \lambda_s^S v_s \end{pmatrix} \right)_{s \in a} \quad \left(im \begin{pmatrix} v_s \\ \vdots \\ \lambda_s^S v_s \end{pmatrix} \right)_{s \in a} \quad \begin{matrix} p \\ \vdots \\ p \end{matrix} \right)$$

has rank $|a| + 1$.

Theorem 3. *For all $(p,(w_i),(r^s))$, an open and dense set of collections of individual utility functions (u_i) and all $(\lambda_s, V_s) \in \mathcal{S}(p)$, there exists a golden rule steady state $(p,(\omega_i))$ in the restricted fiber $F(p,(w_i),(r^s))$ such that for all s*

$$D_q\Gamma(q,(\omega_i))V_s - \lambda_s V_s \;=\; 0.$$

Proof: Only the case, where $\lambda_s \in \mathbb{R}$ for all s, is covered. However, the case where $\lambda_s \in \mathbb{C} \setminus R$ for some s, may be covered by splitting in real parts and imaginary part.

Let the $((2S\ell-1)\ell+m-1) \times ((S+1)\ell(m-1))$-matrix A be defined by

$$A \;=\; \begin{pmatrix} \lambda_1^S A_1^1 & \cdots & A_1^1 & \cdots & \lambda_1^S A_{m-1}^1 & \cdots & A_{m-1}^1 \\ \vdots & & \vdots & & \vdots & & \vdots \\ \lambda_{2S\ell-1}^S A_1^{2S\ell-1} & \cdots & A_1^{2S\ell-1} & \cdots & \lambda_{2S\ell-1}^S A_{m-1}^{2S\ell-1} & \cdots & A_{m-1}^{2S\ell-1} \\ p' & \cdots & p' & & & & \\ & & & & & & \\ & & & & p' & \cdots & p' \end{pmatrix}$$

where the $\ell \times \ell$-matrix A_i^s is defined by

$$A_i^s \;=\; \sum_j \lambda_s^j (D_{w_i} f_i^j(p,\cdots,p,w_i) - D_{w_m} f_m^j(p,\cdots,p,w_m))v_s'$$

and let the $((2S\ell-1)\ell+m-1)$-vector B be defined by

$$B \;=\; \begin{pmatrix} -\lambda_1^S B^1 \\ \vdots \\ -\lambda_{2S\ell-1}^S B^{2S\ell-1} \\ w_1 \\ \vdots \\ w_{m-1} \end{pmatrix}$$

where the ℓ-vector B^s is defined by

$$\begin{aligned} B^s &= \sum_{j,k} \lambda_s^{j-k} \sum_i D_{p_k} f_i^j(p, \ldots, p, w_i) v_s \\ &\quad + \sum_{j,k} \lambda_s^{j-k} D_{w_m} f_m^j(p, \ldots, p, w_m) \sum_i f_i^k(p, \ldots, p, w_i)' v_s \\ &\quad - \sum_{j,k} \lambda_s^{j-k} D_{w_m} f_m^j(p, \ldots, p, w_m) r^{k\prime} v_s. \end{aligned}$$

If the $((S+1)\ell(m-1))$-vector ω is defined by

$$\omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{m-1} \end{pmatrix}$$

then for all s, (λ_s, v_s), where $\lambda_{2S\ell} = 1$ and $v_{2S\ell} = p$, solves

$$\sum_j \lambda_s^{S+j} K_j v_s = 0$$

if and only if

$$A\omega = B.$$

From the proof of Theorem 1 in Ghiglino & Tvede (1997) it follows that for an open and dense set of collections of individual utility functions, all s and all subsets b of $\{1, \ldots, m-1\}$, where $|b| \leq l-1$, the $\ell \times |b|$-matrix

$$((D_{w_i} f_i^s(p, \cdots, p, w_i) - D_{w_m} f_m^s(p, \cdots, p, w_m))_{i \in b})$$

has rank $|b|$. Then, according to condition (B.3) and some tedious calculations, the matrix A has rank $(2S\ell - 1)\ell + m - 1$. Therefore there exists ω such that

$$A\omega = B.$$

From the proof of Theorem 1 in Ghiglino & Tvede (1997) it follows that for an open and dense set of collections of individual utility functions, the matrix K_s has rank ℓ.

Q.E.D

6 On fluctuations

The fact that $\lambda = 1$ is an eigenvalue of $D_q\Gamma(q, (\omega_i))$ for all golden rule steady states $(p, (\omega_i)) \in G(p, (w_i))$ complicates the application of bifurcation theory to the difference equation $q_{t+1}\Gamma(q_t, (\omega_i))$. Therefore, let $\mathbb{S} = \{q \in \mathbb{R}^{2S\ell-1} | \|q\| = 1\}$ and $\mathbb{S}_{++} = \mathbb{S} \cap \mathbb{R}^{2S\ell-1}_{++}$. Then for

$$\begin{aligned} A &= \{\theta \in \mathbb{S}_{++} | \exists q \in U : \frac{1}{\|q\|} q = \theta\} \\ B &= \{\theta \in \mathbb{S}_{++} | \exists q \in V : \frac{1}{\|q\|} q = \theta\}, \end{aligned}$$

the function $\Delta = (\Delta_\theta, \Delta_\gamma) : A \times \mathbb{R}_{++} \times \Omega \to B \times \mathbb{R}_{++}$, where

$$\Delta(\theta, \gamma, (\omega_i)) = \left(\frac{1}{\|\Gamma(\theta, (\omega_i))\|} \Gamma(\theta, (\omega_i)), \|\Gamma(\theta, (\omega_i))\| \gamma\right)$$

is a reformulation of $\Gamma : U \times \Omega \to V$ in polar coordinates. Clearly for $(p, (\omega_i)) \in G(p, (w_i))$ if $(\lambda_1, \ldots, \lambda_{2S\ell})$, where $\lambda_s \in \mathbb{C}$ and $\lambda_{2S\ell} = 1$, is the collection of eigenvalues of $D_q\Gamma(q, (\omega_i))$, then $(\lambda_1, \ldots, \lambda_{2S\ell-1})$ is the collection of eigenvalues of $D_\theta\Delta_\theta(\theta, (\omega_i))$ and $\lambda_{2S\ell} = 1$ is the eigenvalue of $D_\gamma\Delta_\gamma(\theta, \gamma, (\omega_i))$.

According to Theorem 3, the matrix $D_\theta\Delta_\theta$ may have almost all collections of eigenvalues on restricted fibers. Therefore, both the period doubling bifurcation (associated with minus one as eigenvalue) and the Hopf bifurcation (associated with a conjugate pair of complex numbers with modulus one as eigenvalues) may be applied to Δ_θ to establish the existence of periodic and aperiodic fluctuations.

From an economic point of view, fluctuations are of interest because they can be seen as business cycles.

References

Balasko, Y, *The foundation of the theory of general equilibrium* , Academic Press (1988).

Ghiglino, C., & M. Tvede, Endowments, Stability and fluctuations in OG models, Journal of Economic Dynamics and Control **19** (1995), 621-653.

Ghiglino, C., & M. Tvede, Multiplicity of equilibria, Journal of Economic Theory **75** (1997), 1-15.

Kehoe, T. & P. Levine, Regularity in overlapping-generations exchange economies, Journal of Mathematical Economics **13** (1984), 69-93.

Samuelson, P., An exact consumption-loan model with or without the social contrivance of money, Journal of Political Economy **66** (1958), 467-482.