On Excess Differencing in Discrete Time Representations of Cointegrated Continuous Time Models with Mixed Sample*

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Abstract: This paper investigates the source of apparent excess differencing that has been observed to occur in discrete time representations corresponding to cointegrated continuous time systems with a mixed sample of stock and flow variables. Two methods of deriving exact discrete time models are considered and excess differencing is found to manifest itself in both, and in each case the source of the differencing is identified. It is also shown that the excess differencing does not arise in models containing only stock variables or only flow variables. Some further analytical and computational results that enable Gaussian estimation to be implemented are also provided.

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1. INTRODUCTION

To his students and colleagues Rex Bergstrom's passion for economics and econometrics was clearly evident. Nowhere was this more apparent than when he was discussing continuous time methods, an area in which he worked extensively for much of his career and for which he is perhaps best known. Indeed, he continued working on his continuous time macroeconomic model of the UK and the related econometric techniques beyond his retirement in 1992, and it is an ensuing article, Bergstrom (1997), that provides the motivation for the present paper.

The model considered in Bergstrom (1997) is a system of mixed first- and second-order stochastic differential equations with unobservable stochastic trends and mixed stock and flow variables, and it corresponds to the form of the macroeconometric model of the UK developed in Bergstrom and Nowman (2006). The model allows for cointegration among the variables in the system and its discrete time representation is notable for the fact that it is written in terms of the first-differences of the observations. At first sight the discrete time model therefore appears to correspond to a system in which all variables are integrated of order one and not cointegrated, but Bergstrom (2006) provides an example which demonstrates that, in a model with unobservable stochastic trends, the value of the Gaussian likelihood function is unaffected whether the entire system is differenced or whether it is differenced only in directions orthogonal to the cointegration vectors. Since Gaussian estimation was the motivation for the derivation of the exact discrete time model such excess differencing was therefore shown to be unimportant, but Bergstrom (1997, p.485) did acknowledge that a study of the asymptotic sampling properties of the Gaussian estimator would be aided by the derivation of a discrete time model that more closely resembles cointegrated systems formulated directly in discrete time.

This paper picks up the issue of excess differencing in cointegrated continuous time models and attempts to identify its source. The model considered is a system of first-order stochastic differential equations with mixed stock and flow variables in which the system matrix displays the reduced rank implied by cointegration. Two solution methods are analysed and each one is found to suffer from excess differencing. The first method derives the discrete time representation for the vector of observed variables directly, while the second treats the stationary linear combinations of the variables (the cointegrating vectors) separately from the orthogonal combinations. The source of the excess differencing is identified in each case, and it is shown that this phenomenon arises *only* in systems containing mixed stock and flow variables. Indeed, when only stock variables or only flow variables are present, the discrete time representation takes the form of a discrete time error correction model, but the presence of the mixed sample results in an autoregressive representation in first differences. Also derived are the autocovariance matrices of the discrete time disturbance vectors that are utilised in specifiying the time domain Gaussian likelihood function, and some computational issues are briefly discussed as an aid to the implementation of the formulae derived.

It is perhaps also worth mentioning that alternative approaches exist for handling cointegrated continuous time models that do not rely on the exact discrete time model of the type derived in Bergstrom (1997) and that do not display the excess differencing. Harvey and Stock (1988) propose Kalman filter techniques for handling systems with common stochastic trends which avoid the need to derive an exact discrete model for the observations and which can be used to obtain Gaussian estimates. Exact discrete time models in the form of first-order error correction models have been derived by Phillips (1991), Chambers (2003) and Chambers and McCrorie (2007), where in each of these cases the dynamics, potentially generated by higher-order systems, are assigned to the disturbance vector. Phillips suggests frequency domain regression techniques for the estimation of the cointegrating vectors; Chambers uses the representation in a theoretical analysis of the asymptotic efficiency of optimal estimators; and Chambers and McCrorie use frequency domain techniques to approximate the likelihood function and also derive the asymptotic properties of the resulting estimators for the parameters of the cointegrating vectors as well as those governing the dynamics. These methods should be viewed as complementary to those outlined here.

2. THE MODEL

To examine the issues raised above we shall consider the following first-order model

$$dy(t) = [a + bt + Ay(t)]\, dt + \zeta(dt), \quad t > 0 \tag{1}$$

where $y(t)$ is an $n \times 1$ vector of continuous time random variables, $\zeta(dt)$ is an $n \times 1$ vector of random measures satisfying $E[\zeta(dt)] = 0$, $E[\zeta(dt)\zeta(dt)'] = \Sigma dt$ and $E[\zeta(\Delta_1)\zeta(\Delta_2)'] = 0$ for disjoint intervals Δ_1 and Δ_2, a and b are $n \times 1$ vectors of constants, and $A = \alpha\beta'$ where α and β are $n \times r$ matrices of rank $0 < r < n$. The r cointegrating relationships are therefore depicted by the stationary $r \times 1$ vector $\beta' y(t)$. In circumstances in which the cointegrating relationships themselves contain intercepts and deterministic trend terms, the vectors a and b can be restricted appropriately; details of how this is achieved in discrete time cointegrated systems can be found in Pesaran, Shin and Smith (2000) and carry over straightforwardly to the continuous time model. The vector $y(t)$ shall be assumed to comprise both stock and flow variables and, without loss of generality, we shall write $y(t) = [y^s(t)', y^f(t)']'$, where $y^s(t)$ is an $n^s \times 1$ vector of stock variables, $y^f(t)$ is an $n^f \times 1$ vector of flow variables, and $n^s + n^f = n$. Note that the first n^s elements of the initial state vector $y(0)$ are observed and equal to $y^s(0)$, while $y^f(0)$ is unknown.

The observations on the stock variables are of the form $y_t^s = y^s(t)$ $(t = 0, 1, \ldots, T)$ while those on the flow variables are of the form $y_t^f = \int_{t-1}^t y^f(r)dr$ $(t = 1, \ldots, T)$, T denoting sample size. The objective is to relate the unknown parameters of the model (1) to the discrete time observations, which shall be arranged in the $n \times 1$ vector $y_t = [y_t^{s\prime}, y_t^{f\prime}]'$. Note that $w_t^f = y^f(t)$ is unobservable, as is $w_t^s = \int_{t-1}^t y^s(r)dr$, and it is convenient to arrange these components in the unobservable $n \times 1$ vector $w_t = [w_t^{s\prime}, w_t^{f\prime}]'$. As we shall see below the key to deriving discrete time representations that can be used for estimation is the ability to solve out the elements of the unobservable vector w_t from the difference equations that are implied by the solution to (1), which is given by

$$y(t) = e^{tA}y(0) + \int_0^t e^{(t-s)A}(a + bs)ds + \int_0^t e^{(t-s)A}\zeta(ds), \quad t > 0, \tag{2}$$

where $e^{tA} = \sum_{j=0}^{\infty}(tA)^j/j!$ denotes the matrix exponential. It can be shown (see Appendix C) that e^{tA} can be expressed more simply in terms of the exponential of $B = \beta'\alpha$, given by

$$e^{tA} = I_n + \alpha B^{-1}\left(e^{tB} - I_r\right)\beta', \tag{3}$$

where I_n denotes the $n \times n$ identity matrix. It is clear from this representation that the matrix $e^{tA} - I_n$ retains the reduced rank of A.

3. METHOD 1

There are three key equations that motivate this method. The first two are implications of the solution of the model, given by (2), from which it follows, defining $J = e^A$, that

$$y(t) = Jy(t-1) + \mu + \gamma t + \eta_t, \quad t = 1, \ldots, T, \tag{4}$$

where $\mu = Ga - Hb$, $\gamma = Gb$, $G = \int_0^1 e^{sA}ds$, $H = \int_0^1 se^{sA}ds$, and $\eta_t = \int_{t-1}^t e^{(t-s)A}\zeta(ds)$. Integrating (4) over the interval $(t-1, t]$ yields

$$\int_{t-1}^t y(r)dr = J\int_{t-2}^{t-1} y(r)dr + m + \gamma t + v_t, \quad t = 1, \ldots, T, \tag{5}$$

where $m = \mu - (\gamma/2)$ and $v_t = \int_{t-1}^t \int_{r-1}^r e^{(r-s)A}\zeta(ds)dr$. Finally, integrating the model (1) over $(t-1, t]$ yields

$$\Delta y(t) = g + bt + A\int_{t-1}^t y(r)dr + e_t, \tag{6}$$

where $g = a - (b/2)$ and $e_t = \int_{t-1}^t \zeta(dr)$. These three equations can be used to derive the following exact discrete time representation, in which vectors and matrices are partitioned conformably with the stock and flow components (so that, for example, the vector α_f is $n^f \times 1$ and the matrix J_{sf} is $n^s \times n^f$).

THEOREM 1. *Let $y(t)$ be generated by (1). Then, if $n^s \geq r$, the observable vector y_t*

satisfies

$$
\begin{aligned}
y_1 &= \Phi_0 y(0) + \phi_0 + u_1, & (7)\\
\Delta y_t &= \Phi \Delta y_{t-1} + \phi_1 + \phi_2 t + u_t, \quad t = 2, \ldots, T, & (8)
\end{aligned}
$$

where the parameter matrices and disturbance vectors in (8) and (7) are defined by

$$\Phi = \begin{pmatrix} J_{ss} + J_{sf}\alpha_f P_s & 0 \\ \alpha_f F P_s & 0 \end{pmatrix}, \; \Phi_0 = \begin{pmatrix} J_{ss} & J_{sf} \\ G_{fs} & G_{ff} \end{pmatrix}, \; P_s = (\alpha_s'\alpha_s)^{-1}\alpha_s', \; F = B^{-1}(e^B - I_r),$$

$$\phi_0 = \begin{pmatrix} \mu_s + \gamma_s \\ m_0^f \end{pmatrix}, \; \phi_1 = \begin{pmatrix} \gamma_s - J_{sf}\left[b_f + \alpha_f P_s(g_s - b_s)\right] \\ m_f - \alpha_f F P_s(g_s - b_s) \end{pmatrix}, \; \phi_2 = \begin{pmatrix} J_{sf}(b_f - \alpha_f P_s b_s) \\ \gamma_f - \alpha_f F P_s b_s \end{pmatrix},$$

$$m_0 = G_0 a + H_0 b, \quad G_0 = \int_0^1 \int_0^r e^{(r-s)A} ds dr, \quad H_0 = \int_0^1 \int_0^r r e^{(r-s)A} ds dr,$$

$$u_t^s = \Delta\eta_t^s + J_{sf}\left(e_{t-1}^f - \alpha_f P_s e_{t-1}^s\right), \quad (t = 2, \ldots, T), \quad u_1^s = \eta_1^s,$$

$$u_t^f = v_t^f - \alpha_f F P_s e_{t-1}^s, \quad (t = 2, \ldots, T), \quad u_1^f = \rho_1^f, \quad \rho_1 = \int_0^1 \int_0^r e^{(r-s)A}\zeta(ds) dr.$$

Remark 1. The system of equations in (8) is entirely in terms of the first differences Δy_t. This is somewhat unusual in view of the cointegration in the system, but is entirely in accordance with the results of Bergstrom (1997).

Remark 2. Another interesting feature of the system (8) is that it is entirely driven by lagged values of Δy_t^s but not of Δy_t^f. A possible explanation for this is that Δy_t^s contains, up to a white noise vector, information about the cointegrating relationship $\beta' \int_{t-1}^t y(r)dr$. To see this, the first n^s equations of (6) give

$$\Delta y_t^s = g_s + b_s t + \alpha_s \left(\beta_s' w_t^s + \beta_f' y_t^f\right) + e_t^s, \tag{9}$$

and hence the presence of Δy_{t-1}^s as the driving force in the system is acting as a proxy for the partially unobservable cointegration term $\beta_s' w_{t-1}^s + \beta_f' y_{t-1}^f$. Indeed, (6) appears to be the source of the excess differencing in the discrete time model.

Remark 3. The condition $n^s \geq r$ is needed so that the matrix P_s can be computed. It enables (9) to be solved for $\beta_s' w_t^s + \beta_f' y_t^f$ in terms of Δy_t^s.

Remark 4. When the sample consists entirely of stock variables the exact discrete time model is provided directly by (4). Noting, from (3), that $J = I_n + \alpha F \beta'$, it is clear that in this case $y(t)$ satisfies the error correction model (ECM)

$$\Delta y(t) = \alpha F \beta' y(t-1) + \mu + \gamma t + \eta_t,$$

where η_t is vector white noise. Similarly, in the case of a sample comprised entirely of flow variables, (5) yields the ECM

$$\Delta y_t = \alpha F \beta' y_{t-1} + m + \gamma t + v_t,$$

in which v_t is a vector MA(1) process. This emphasises the fact that it is essentially the presence of a mixed sample that results in the form of the discrete time model in (8).

4. METHOD 2

An alternative approach is to derive an exact discrete time model for the r stationary linear combinations $\beta' y_t$ as well as the $n-r$ linear combinations of the differences Δy_t in orthogonal directions to the cointegrating space. These latter combinations will be denoted $\beta_\perp' \Delta y_t$, where $\beta_\perp$ in an $n \times (n-r)$ matrix satisfying $\beta_\perp' \beta = 0$. We shall also assume that β and $\beta_\perp'$ are normalised so that $\beta'\beta = I_r$, $\beta_\perp' \beta_\perp = I_{n-r}$ and $\beta\beta' + \beta_\perp \beta_\perp' = I_n$. The usual approach in discrete time is to define the matrix $K_d = [\beta, \beta_\perp]$ which satisfies $K_d' K_d = K_d K_d' = I_n$ and to transform the model by pre-multiplying by K_d'. When applied to the continuous time system (1) this yields

$$K_d' dy(t) = [K_d' a + K_d' bt + K_d' A K_d K_d' y(t)]\, dt + K_d' \zeta(dt), \tag{10}$$

which can also be written as the pair of equations

$$d\beta' y(t) = [\beta' a + \beta' bt + \beta' \alpha \beta' y(t)]\, dt + \beta' \zeta(dt), \tag{11}$$

$$\beta_\perp' dy(t) = [\beta_\perp' a + \beta_\perp' bt + \beta_\perp' \alpha \beta' y(t)]\, dt + \beta_\perp' \zeta(dt). \tag{12}$$

The difficulty in working with this representation when there is a mixed sample of stock and flow variables lies in the need to treat the temporal aggregation of the different components of $\beta' y(t)$ and $\beta_\perp' y(t)$ in different ways. In particular, because $\beta' y(t) = \beta_s' y^s(t) + \beta_f' y^f(t)$, its counterpart in terms of observed variables is $\beta' y_t = \beta_s' y_t^s + \beta_f' y_t^f$, which involves elements of $\beta' y(t)$ and $\beta' \int_{t-1}^t y(r)dr$ which can't be extracted separately from each expression.

Our proposed solution to this problem utilises an extended transformation matrix K of dimension $n \times 2n$ defined by

$$K = \begin{pmatrix} \beta_s & 0 & \beta_{s\perp} & 0 \\ 0 & \beta_f & 0 & \beta_{f\perp} \end{pmatrix}$$

and which is normalised so as to satisfy $K'K = I_{2n}$ and $KK' = I_n$. The transformation $K' y(t)$ then enables us to pick out terms such as $\beta_s' y^s(t)$ and $\beta_f' y^f(t)$, as well as their orthogonal counterparts, and treat them separately for the purposes of the temporal aggregation.

It is convenient to define the random vector $z(t) = K'y(t) = [z_1(t)', z_2(t)']'$, where

$$z_1(t) = \begin{pmatrix} \beta_s' y^s(t) \\ \beta_f' y^f(t) \end{pmatrix} \quad (2r \times 1), \quad z_2(t) = \begin{pmatrix} \beta_{s\perp}' y^s(t) \\ \beta_{f\perp}' y^f(t) \end{pmatrix} \quad (2(n-r) \times 1),$$

as well as the matrix $C = K'\alpha\beta' K$ and vectors $\overline{a} = K'a$ and $\overline{b} = K'b$ which are of the form

$$C = \begin{pmatrix} C_1 & 0 \\ C_2 & 0 \end{pmatrix} = \begin{pmatrix} \beta_s'\alpha_s & \beta_s'\alpha_s & 0 & 0 \\ \beta_f'\alpha_f & \beta_f'\alpha_f & 0 & 0 \\ \beta_{s\perp}'\alpha_s & \beta_{s\perp}'\alpha_s & 0 & 0 \\ \beta_{f\perp}'\alpha_s & \beta_{f\perp}'\alpha_s & 0 & 0 \end{pmatrix}, \quad \overline{a} = \begin{pmatrix} \overline{a}_1 \\ \overline{a}_2 \end{pmatrix}, \quad \overline{b} = \begin{pmatrix} \overline{b}_1 \\ \overline{b}_2 \end{pmatrix}.$$

Then, defining $\overline{\zeta}(dt) = K'\zeta(dt)$, the transformed system can be written

$$dz(t) = \left[\overline{a} + \overline{b}t + Cz(t)\right] dt + \overline{\zeta}(dt), \tag{13}$$

whose solution is of the same form as that given in (2) for $y(t)$:

$$z(t) = e^{tC} z(0) + \int_0^t e^{(t-s)C} \left(\overline{a} + \overline{b}s\right) ds + \int_0^t e^{(t-s)C} \overline{\zeta}(ds), \quad t > 0. \tag{14}$$

The solution vector $z(t)$ therefore also satisfies the stochastic difference equation

$$z(t) = e^C z(t-1) + c_t + \epsilon_t, \tag{15}$$

where

$$c_t = \begin{pmatrix} c_{1t} \\ c_{2t} \end{pmatrix} = \int_{t-1}^t e^{(t-s)C} \left(\overline{a} + \overline{b}s\right) ds \quad \text{and} \quad \epsilon_t = \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix} = \int_{t-1}^t e^{(t-s)C} \overline{\zeta}(ds).$$

From the form of the matrix C we find that

$$C^j = \begin{pmatrix} C_1^j & 0 \\ C_2 C_1^{j-1} & 0 \end{pmatrix} \quad \text{and so} \quad e^{sC} = \begin{pmatrix} e^{sC_1} & 0 \\ C_2\Psi_1(s) & I_{2(n-r)} \end{pmatrix},$$

where $\Psi_1(s) = \sum_{j=1}^{\infty} s^j C_1^{j-1}/j\,!$. It is also convenient to define

$$e^C = \begin{pmatrix} \Theta & 0 \\ \Gamma & I_{2(n-r)} \end{pmatrix},$$

where $\Theta = e^{C_1}$ and $\Gamma = C_2\Psi_1(1)$. Furthermore, from the form of the submatrix C_1, it can be shown that

$$\Theta = \begin{pmatrix} \Theta_{ss} & \Theta_{sf} \\ \Theta_{fs} & \Theta_{ff} \end{pmatrix} = \begin{pmatrix} \Theta_{ss} & \Theta_{ss} - I_r \\ \Theta_{ff} - I_r & \Theta_{ff} \end{pmatrix} = \begin{pmatrix} \Theta_{sf} + I_r & \Theta_{sf} \\ \Theta_{fs} & \Theta_{fs} + I_r \end{pmatrix}.$$

This is an important result in what follows and can be traced as the source of the apparent excess differencing in the discrete time model below. Note, too, that the partitions of the matrix Θ are all $r \times r$, corresponding now to the $r \times 1$ components $\beta_s' y_t^s$ and $\beta_f' y_t^f$, rather than the stock and flow vectors themselves, as in section 3. Also, in Theorem 2 below, the subscripts 1 and 2 denote partitions of vectors conformably with z_1 and z_2 and, hence, are $2r \times 1$ and $2(n-r) \times 1$, respectively.

THEOREM 2. *Let $y(t)$ be generated by (1). Then, assuming that $n^s \geq r$, $n^f \geq r$, and that Θ_{sf} and Θ_{fs} are nonsingular, the observable vector y_t satisfies*

$$\beta' y_1 = K_{10}^s \beta_s' y^s(0) + K_{10}^f \beta_f' y^f(0) + \overline{\tau}_{11} + \overline{\xi}_{11}, \quad (16)$$

$$\beta_\perp'[y_1 - y(0)] = N_{10}^s \beta_s' y^s(0) + N_{10}^f \beta_f' y^f(0) + \overline{\tau}_{21} + \overline{\xi}_{21}, \quad (17)$$

$$\beta' y_2 = \Theta_{ss}\beta_s'[y_1^s - y^s(0)] + \Theta_{ff}\beta_f'[y_1^f - y^f(0)] + \overline{K}_{20}\beta' y(0) + \overline{\tau}_{12} + \overline{\xi}_{12}, \quad (18)$$

$$\beta_\perp' \Delta y_2 = \Gamma_{ss}\beta_s'[y_1^s - y^s(0)] + \Gamma_{ff}\beta_f'[y_1^f - y^f(0)] + \overline{N}_{20}\beta' y(0) + \overline{\tau}_{22} + \overline{\xi}_{22}, \quad (19)$$

$$\beta' \Delta y_t = K_2^s \beta_s' \Delta y_{t-1}^s + K_2^f \beta_f' \Delta y_{t-1}^f + \overline{\tau}_{1t} + \overline{\xi}_{1t}, \quad t = 3, \ldots, T, \quad (20)$$

$$\beta_\perp' \Delta y_t = N_1^s \beta_s' \Delta y_{t-1}^s + N_1^f \beta_f' \Delta y_{t-1}^f + \overline{\tau}_{2t} + \overline{\xi}_{2t}, \quad t = 3, \ldots, T, \quad (21)$$

where

$$K_{10}^s = \Theta_{ss} + \Psi_0^{fs}, \quad K_{10}^f = \Theta_{sf} + \Psi_0^{ff}, \quad N_{10}^s = \Gamma_{ss} + \widetilde{\Gamma}_{fs}, \quad N_{10}^f = \Gamma_{sf} + \widetilde{\Gamma}_{ff},$$

$$\overline{K}_{20} = \Theta_{ss}\Theta_{ff} + \Theta_{ff}\Theta_{ss} - \Theta_{ss} - \Theta_{ff} + I_r, \quad \overline{N}_{20} = \Gamma_{ss}\Theta_{ff} + \Gamma_{ff}\Theta_{ss},$$

$$K_2^s = \Theta_{sf}\Theta_{ff}\Theta_{sf}^{-1}\Theta_{ss} - \Theta_{sf}\Theta_{fs}, \quad K_2^f = \Theta_{fs}\Theta_{ss}\Theta_{fs}^{-1}\Theta_{ff} - \Theta_{fs}\Theta_{sf},$$

$$N_1^s = \Gamma_{ss} + \Gamma_{sf}\Theta_{ff}\Theta_{sf}^{-1}, \quad N_1^f = \Gamma_{ff} + \Gamma_{fs}\Theta_{ss}\Theta_{fs}^{-1},$$

$$\Psi_0 = \int_0^1 e^{rC_1} dr, \quad \widetilde{\Gamma} = C_2\Psi_2(1), \quad \Psi_2(r) = \int_0^r \Psi_1(s)ds,$$

$$\overline{\tau}_{11} = c_{11}^s + \widetilde{\psi}_{11}^f, \quad \overline{\tau}_{12} = c_{12}^s + \Theta_{sf}c_{11}^f + \psi_{12}^f + \Theta_{fs}\psi_{11}^s,$$

$$\overline{\tau}_{21} = c_{21}^s + \widetilde{\psi}_{21}^f, \quad \overline{\tau}_{22} = c_{22}^s + \Gamma_{sf}c_{11}^f + \psi_{22}^f + \Gamma_{fs}\psi_{11}^s,$$

$$\overline{\tau}_{1t} = \tau_{1t}^s + \tau_{1t}^f, \quad \overline{\tau}_{2t} = \Gamma_{sf}\Theta_{sf}^{-1}\tau_{1t}^s + \Gamma_{fs}\Theta_{fs}^{-1}\tau_{1t}^f + \tau_{2t}^s + \tau_{2t}^f, \quad t = 3, \ldots, T,$$

$$\tau_{1t}^s = c_{1t}^s - \Theta_{sf}\Theta_{ff}\Theta_{sf}^{-1}c_{1,t-1}^s + \Theta_{sf}c_{1,t-1}^f, \quad \tau_{2t}^s = c_{2t}^s - \Gamma_{sf}\Theta_{sf}^{-1}c_{1t}^s, \quad t = 3, \ldots, T,$$

$$\tau_{1t}^f = \psi_{1t}^f - \Theta_{fs}\Theta_{ss}\Theta_{fs}^{-1}\psi_{1,t-1}^f + \Theta_{fs}\psi_{1,t-1}^s, \quad \tau_{2t}^f = \psi_{2t}^f - \Gamma_{fs}\Theta_{fs}^{-1}\psi_{1t}^f, \quad t = 3, \ldots, T,$$

$$\widetilde{\psi}_1 = \begin{pmatrix} \widetilde{\psi}_{11} \\ \widetilde{\psi}_{21} \end{pmatrix} = \int_0^1 \int_0^r e^{(r-s)C}\left(\overline{a} + \overline{b}s\right) ds dr,$$

$$\psi_t = \begin{pmatrix} \psi_{1t} \\ \psi_{2t} \end{pmatrix} = \int_{t-1}^t \int_{r-1}^r e^{(r-s)C}\left(\overline{a} + \overline{b}s\right) ds dr, \quad t = 2, \ldots, T,$$

$$\overline{\xi}_{11} = \epsilon_{11}^s + \widetilde{\nu}_{11}^f, \quad \overline{\xi}_{12} = \epsilon_{12}^s + \Theta_{sf}\epsilon_{11}^f + \nu_{12}^f + \Theta_{fs}\nu_{11}^s,$$

$$\overline{\xi}_{21} = \epsilon_{21}^s + \widetilde{\nu}_{21}^f, \quad \overline{\xi}_{22} = \epsilon_{22}^s + \Gamma_{sf}\epsilon_{11}^f + \nu_{22}^f + \Gamma_{fs}\nu_{11}^s,$$

$$\overline{\xi}_{1t} = \xi_{1t}^s + \xi_{1t}^f, \quad \overline{\xi}_{2t} = \Gamma_{sf}\Theta_{sf}^{-1}\xi_{1t}^s + \Gamma_{fs}\Theta_{fs}^{-1}\xi_{1t}^f + \xi_{2t}^s + \xi_{2t}^f, \quad t = 3, \ldots, T,$$

$$\xi_{1t}^s = \epsilon_{1t}^s - \Theta_{sf}\Theta_{ff}\Theta_{sf}^{-1}\epsilon_{1,t-1}^s + \Theta_{sf}\epsilon_{1,t-1}^f, \quad \xi_{2t}^s = \epsilon_{2t}^s - \Gamma_{sf}\Theta_{sf}^{-1}\epsilon_{1t}^s, \quad t = 3, \ldots, T,$$

$$\xi_{1t}^f = \nu_{1t}^f - \Theta_{fs}\Theta_{ss}\Theta_{fs}^{-1}\nu_{1,t-1}^f + \Theta_{fs}\nu_{1,t-1}^s, \quad \xi_{2t}^f = \nu_{2t}^f - \Gamma_{fs}\Theta_{fs}^{-1}\nu_{1t}^f, \quad t = 3, \ldots, T,$$

$$\widetilde{\nu}_1 = \begin{pmatrix} \widetilde{\nu}_{11} \\ \widetilde{\nu}_{21} \end{pmatrix} = \int_0^1 \int_0^r e^{(r-s)C}\overline{\zeta}(ds)dr,$$

$$\nu_t = \begin{pmatrix} \nu_{1t} \\ \nu_{2t} \end{pmatrix} = \int_{t-1}^t \int_{r-1}^r e^{(r-s)C}\overline{\zeta}(ds)dr, \ t = 2, \ldots, T.$$

Remark 1. The representation in (20) appears to be over-differenced in that $\beta'\Delta y_t$, rather than $\beta' y_t$, is on the left-hand-side. Once again this would seem to be due to the simultaneous presence of both stock and flow variables. If the sample comprised only stock variables, so that $\beta' y_t = \beta_s' y_t^s$, then (34) and (35) in Appendix A provide the exact discrete time model for $t = 1, \ldots, T$:

$$\beta' y_t = \Theta\beta' y_{t-1} + c_{1t} + \epsilon_{1t}, \quad \beta_\perp' \Delta y_t = \Gamma\beta' y_{t-1} + c_{2t} + \epsilon_{2t}.$$

On the other hand, if the sample consisted entirely of flow variables, so that $\beta' y_t = \beta_f' y_t^f$, then (43) and (44) in Appendix A describe the exact discrete time model for $t = 2, \ldots, T$:

$$\beta' y_t = \Theta\beta' y_{t-1} + \psi_{1t} + \nu_{1t}, \quad \beta_\perp' \Delta y_t = \Gamma\beta' y_{t-1} + \psi_{2t} + \nu_{2t}.$$

Neither of these representations requires the stationary linear combination $\beta' y_t$ to be differenced, emphasising that it is the necessity of treating the stock and flow components differently in a mixed sample that results in the differencing of this variable.

Remark 2. The differences of the lagged stock and flow variables on the right-hand-sides of (20) and (21) appear separately rather than as the single term $\beta'\Delta y_{t-1}$. This is yet another manifestation of the fact that the variables are, by necessity, treated differently in the derivation of the discrete time model. In the proof of Theorem 2 in Appendix A it can be seen in (48) and (53) that two seemingly more natural representations occur, which are

$$\beta' y_t = K_1^s \beta_s' y_{t-1}^s + K_1^f \beta_f' y_{t-1}^f - K_2^s \beta_s' y_{t-2}^s - K_2^f \beta_f' y_{t-2}^f + \overline{\tau}_{1t} + \overline{\xi}_{1t},$$

$$\beta_\perp' \Delta y_t = N_1^s \beta_s' y_{t-1}^s + N_1^f \beta_f' y_{t-1}^f + N_2^s \beta_s' y_{t-2}^s + N_2^f \beta_f' y_{t-2}^f + \overline{\tau}_{2t} + \overline{\xi}_{2t}.$$

The lagged terms on the right-hand-sides of these equations, however, are not necessarily stationary, because the stationarity of $\beta' y_t$ does not imply the stationarity of $\beta_s' y_t^s$ and $\beta_f' y_t^f$ themselves. The differences of these components that appear in Theorem 2 arise because of the relationships between the coefficient matrices depicted in Proposition 1 in Appendix C and ensure the stationarity of the individual terms in the resulting discrete time representation.

Remark 3. The source of the apparent excess differencing in the discrete time representation can be traced to the relationship between the submatrices of the matrix $\Theta = e^{C_1}$.

These have an impact most notably in transforming the seemingly natural representation for $\beta' y_t$ in (48), reproduced above, into the first-differenced representation (20). The form of the matrix e^{C_1} itself arises due to the transformation from continuous time to discrete time and the necessity to treat stock variables and flow variables differently; the apparent excess differencing does not arise with stocks or flows alone (see Remark 1).

Remark 4. The conditions $n^s \geq r$ and $n^f \geq r$ arise so that an appropriately normalised transformation matrix K can be defined. In the usual discrete time case it is straightforward to take $K_d = [\beta, \beta_\perp]$ where, if β^* denotes the underlying matrix of cointegrating parameters of interest, it is possible to take $\beta = \beta^*(\beta^{*\prime}\beta^*)^{-1/2}$ (because $n > r$) which satisfies $\beta'\beta = I_r$. In the situation here it is necessary to treat β_s and β_f separately, and such a normalisation is not feasible if $n^s < r$ or $n^f < r$, both of which are possible while still retaining $n > r$.

Remark 5. The increments $y_1^s - y^s(0)$ and $y_1^f - y^f(0)$ that appear in (17), (18) and (19) are stationary (net of trend). This can be seen by noting that $\Theta_{ss} = \Theta_{sf} + I_r$ and $\Psi_0^{ff} = \Psi_0^{fs} + I_r$ and substituting into (54) and (57), respectively, in Appendix A to give

$$\beta_s' \left[y_1^s - y^s(0)\right] = \Theta_{sf}\beta' y(0) + c_{11}^s + \epsilon_{11}^s, \quad \beta_f' \left[y_1^f - y^f(0)\right] = \Psi_0^{fs}\beta' y(0) + \widetilde{\psi}_1^f + \widetilde{\nu}_{11}^f.$$

5. GAUSSIAN ESTIMATION AND AUTOCOVARIANCES

Estimates of the unknown parameters of the model (1) can be obtained by maximising the Gaussian likelihood function. In the case of Method 1 it is convenient to define the $nT \times 1$ vector $u = (u_1', \ldots, u_T')'$ and its $nT \times nT$ block-diagonal Toeplitz covariance matrix $\Omega = E(uu')$. Assuming that u is multivariate Gaussian (which is equivalent to assuming that $\zeta(dt)$ is Gaussian or, equivalently, the increment of a Brownian motion process), minus twice the negative of the logarithm of the Gaussian likelihood function can be written (ignoring the constant term)

$$\ln L_1 = \ln |\Omega| + u'\Omega^{-1}u.$$

In order for this approach to be followed it is therefore necessary to derive expressions for the autocovariances of the discrete time disturbance vectors that define the matrix Ω. This is facilitated by first reducing the expressions involving double integrals of the random measure $\zeta(dt)$ to single integrals using the approach of Bergstrom (1997) and McCrorie (2000); details can be found in Appendix B. Theorem 3 provides the autocovariances of the discrete time disturbance vector u_t in Method 1.

THEOREM 3. *Let u_t be defined as in Theorem 1. Then*

$$E(u_1 u_1') = \Omega_{11} = \int_0^1 J_0(s)\Sigma J_0(s)' ds,$$

$$E(u_t u_t') = \Omega_0 = \int_0^1 J_0(s)\Sigma J_0(s)'ds + \int_0^1 J_1(s)\Sigma J_1(s)'ds, \quad t = 2, \ldots, T,$$

$$E(u_t u_{t-1}') = \Omega_1 = \int_0^1 J_1(s)\Sigma J_0(s)'ds, \quad t = 2, \ldots, T,$$

where $J_0(s) = S_1 F_0(s) + S_2 F_1(s),\ J_1(s) = S_2 F_2(s) - S_1 F_0(s) + S_3,$

$$S_1 = \begin{pmatrix} I_{n^s} & 0 \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n^f} \end{pmatrix}, \quad S_3 = \begin{pmatrix} -J_{sf}\alpha_f P_s & J_{sf} \\ -\alpha_f F P_s & 0 \end{pmatrix},$$

$$F_0(s) = e^{sA}, \quad F_1(s) = \int_0^s e^{rA}dr, \quad F_2(s) = \int_s^1 e^{rA}dr.$$

The disturbance vector u_t is therefore a vector MA(1) process implying that Δy_t in (8) is a vector ARMA(1,1) process. The autocovariances for the discrete time disturbances in Method 2 are provided in Theorem 4.

THEOREM 4. *Let* $\bar{\xi}_t = (\bar{\xi}_{1t}', \bar{\xi}_{2t}')'$, $t = 1, \ldots, T$. *Then*

$$E(\bar{\xi}_1\bar{\xi}_1') = \int_0^1 W_0(s)\Sigma W_0(s)'ds,$$

$$E(\bar{\xi}_2\bar{\xi}_1') = \int_0^1 W_3(s)\Sigma W_0(s)'ds,$$

$$E(\bar{\xi}_2\bar{\xi}_2') = \int_0^1 W_0(s)\Sigma W_0(s)'ds + \int_0^1 W_3(s)\Sigma W_3(s)'ds,$$

$$E(\bar{\xi}_3\bar{\xi}_1') = \int_0^1 W_2(s)\Sigma W_0(s)'ds,$$

$$E(\bar{\xi}_3\bar{\xi}_2') = \int_0^1 W_1(s)\Sigma W_0(s)'ds + \int_0^1 W_2(s)\Sigma W_3(s)'ds,$$

$$E(\bar{\xi}_t\bar{\xi}_t') = \int_0^1 W_0(s)\Sigma W_0(s)'ds + \int_0^1 W_1(s)\Sigma W_1(s)'ds + \int_0^1 W_2(s)\Sigma W_2(s)'ds, \quad t = 3, \ldots, T,$$

$$E(\bar{\xi}_t\bar{\xi}_{t-1}') = \int_0^1 W_1(s)\Sigma W_0(s)'ds + \int_0^1 W_2(s)\Sigma W_1(s)'ds, \quad t = 4, \ldots, T,$$

$$E(\bar{\xi}_t\bar{\xi}_{t-2}') = \int_0^1 W_2(s)\Sigma W_0(s)'ds, \quad t = 4, \ldots, T,$$

where $W_0(s) = S_4 R_0(s) + S_6 R_1(s)$, $W_1(s) = S_5 R_0(s) + S_6 R_2(s) + S_7 R_1(s)$, $W_2(s) = S_7 R_2(s)$, $W_3(s) = S_8 R_0(s) + S_6 R_2(s) + S_9 R_1(s)$,

$$S_4 = \begin{pmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & I_{n-r} & 0 \end{pmatrix}, \quad S_5 = \begin{pmatrix} -\Theta_{sf}\Theta_{ff}\Theta_{sf}^{-1} & \Theta_{sf} & 0 & 0 \\ -\Gamma_{sf}\Theta_{ff}\Theta_{sf}^{-1} & \Gamma_{sf} & 0 & 0 \end{pmatrix},$$

$$S_6 = \begin{pmatrix} 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & I_{n-r} \end{pmatrix}, \quad S_7 = \begin{pmatrix} \Theta_{fs} & -\Theta_{fs}\Theta_{ss}\Theta_{fs}^{-1} & 0 & 0 \\ \Gamma_{fs} & -\Gamma_{fs}\Theta_{ss}\Theta_{fs}^{-1} & 0 & 0 \end{pmatrix},$$

$$S_8 = \begin{pmatrix} 0 & \Theta_{sf} & 0 & 0 \\ 0 & \Gamma_{sf} & 0 & 0 \end{pmatrix}, \quad S_9 = \begin{pmatrix} \Theta_{fs} & 0 & 0 & 0 \\ \Gamma_{fs} & 0 & 0 & 0 \end{pmatrix},$$

$$R_0(s)=\begin{pmatrix} e^{sC_1} & 0 \\ C_2\Psi_1(s) & I_{n-r}\end{pmatrix}K', \quad R_1(s)=\begin{pmatrix} \Psi_3(s) & 0 \\ C_2\Psi_2(s) & s\end{pmatrix}K',$$

$$R_2(s)=\begin{pmatrix} \Psi_3(1)-\Psi_3(s) & 0 \\ -C_2\Psi_2(s) & s-1\end{pmatrix}K', \quad \Psi_3(s)=\int_0^s e^{rC_1}dr.$$

Theorem 4 shows that the vector $\overline{\xi}_t$ is an MA(2) process. Defining $\overline{\xi}=(\overline{\xi}_1',\ldots,\overline{\xi}_T')'$ and $\overline{\Omega}=E(\overline{\xi}\overline{\xi}')$ the objective function for Gaussian estimation can be written

$\ln L_2=\ln|\overline{\Omega}|+\overline{\xi}'\overline{\Omega}^{-1}\overline{\xi}.$

Although we have derived expressions that determine the components needed for Gaussian estimation using either $\ln L_1$ or $\ln L_2$, some further investigation of these expressions is necessary from a practical computational point of view.

6. COMPUTATIONAL ISSUES

Inspection of the formulae in Theorems 1 and 2 reveals that a number of the matrices and vectors appear as integrals involving the matrix exponential and other functions. The autocovariances used in computing the Gaussian likelihood functions $\ln L_1$ and $\ln L_2$ also involve integrals of matrix exponentials. Theorem 5 contains computable representations for the relevant expressions in Method 1.

THEOREM 5. *The matrices G, H, G_0 and H_0, and functions $J_0(s)$ and $J_1(s)$, in Method 1 have the following representations:*

$$G=\int_0^1 e^{sA}ds=\Pi_1+\Pi_2F\beta', \quad G_0=\int_0^1\int_0^r e^{(r-s)A}dsdr=\frac{1}{2}\Pi_1+\Pi_2B^{-1}\left(F-I_r\right)\beta',$$

$$H=\int_0^1 se^{sA}ds=\frac{1}{2}\Pi_1+\Pi_2\Pi_3\beta', \quad H_0=\int_0^1\int_0^r re^{(r-s)A}dsdr=\frac{1}{3}\Pi_1+\Pi_2B^{-1}\left(\Pi_3-\frac{1}{2}I_r\right)\beta',$$

$$J_i(s)=J_{i0}+J_{i1}s+J_{i2}e^{sB}\beta', \quad i=0,1,$$

where $\Pi_1=I_n-\alpha B^{-1}\beta'$, $\Pi_2=\alpha B^{-1}$, *and* $\Pi_3=B^{-1}(e^B-F)$, $J_{00}=S_1\Pi_1-S_2\Pi_2B^{-1}\beta'$, $J_{01}=S_2\Pi_1$, $J_{02}=S_1\Pi_2+S_2\Pi_2B^{-1}$, $J_{10}=S_2(\Pi_1+\Pi_2B^{-1}e^B\beta')-S_1\Pi_1+S_3$, $J_{11}=-S_2\Pi_1$, *and* $J_{12}=-S_1\Pi_2-S_2\Pi_2B^{-1}$. *The integrals determining the autocovariance matrices* Ω_{11}, Ω_0 *and* Ω_1 *then have the representation*

$$\begin{aligned}\int_0^1 J_i(s)\Sigma J_k(s)'ds &= \int_0^1\left[J_{i0}+J_{i1}s+J_{i2}e^{sB}\beta'\right]\Sigma\left[J_{k0}+J_{k1}s+J_{k2}e^{sB}\beta'\right]'ds \\ &= J_{i0}\Sigma J_{k0}'+\frac{1}{2}J_{i0}\Sigma J_{k1}'+J_{i0}\Sigma\beta F'J_{k2}'+\frac{1}{2}J_{i1}\Sigma J_{k0}'+\frac{1}{3}J_{i1}\Sigma J_{k1}' \\ &\quad +J_{i1}\Sigma\beta\Pi_3'J_{k2}'+J_{i2}F\beta'\Sigma J_{k0}'+J_{i2}\Pi_3\beta'\Sigma J_{k1}'+J_{i2}\Pi_4J_{k2}', \qquad (22)\end{aligned}$$

where $\Pi_4=\int_0^1 e^{sB}\beta'\Sigma\beta e^{sB'}ds$.

The only remaining integral and matrix exponential that require computation are therefore Π_4 and e^B. In fact, both can be obtained from the computation of a single matrix exponential, this being

$$M = \exp\begin{pmatrix} -B & \beta'\Sigma\beta \\ 0 & B' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}.$$

Jewitt and McCrorie (2005), using results of van Loan (1978), show that $\Pi_4 = M_{22}'M_{12}$ and $e^B = M_{22}'$. Hence the problem is reduced to the computation of e^M, and a number of methods exist for this purpose. The most staightforward of these is truncation of the infinite series representation at some sufficiently large integer n, resulting in the approximation

$$e^M \approx (e^M)_n = I_{2r} + \sum_{j=1}^{n} \frac{M^j}{j!}.$$

The integer n can be chosen such that the elements of the difference $(e^M)_n - (e^M)_{n-1}$ are sufficiently small in absolute value. An investigation of alternative methods of computing matrix exponentials in the context of continuous time macroeconometric models can be found in Jewitt and McCrorie (2005), and although they recommend a method based on a trigonometric representation, they conclude that the problem does not appear to be ill-conditioned for the types of models typically considered in the literature so that the truncation method should be adequate.

Method 2 requires similar, but somewhat distinct, considerations, and is slightly more complex than Method 1 from a computational point of view. As no particularly new insights are obtained by presenting the relevant expressions here it will suffice to highlight some differences that arise compared to Method 1. The first of these is that, for some expressions, it appears necessary to truncate infinite series. A leading example is in the computation of the submatrix Γ of e^C which is defined as $\Gamma = C_2\Psi_1(1)$ where $\Psi_1(1) = \sum_{j=1}^{\infty} C_1^{j-1}/j!$ is closely related to the exponential of C_1 by noting that $C_1\Psi_1(1) = e^{C_1} - I_{2r}$. The singularity of C_1 rules out simple inversion and premultiplication, so truncation of the infinite series would appear necessary. This also applies to $\widetilde{\Gamma} = C_2\Psi_2(1)$ where $\Psi_2(1) = \int_0^1 \Psi_1(s)ds$ can be obtained by integrating the expansion of $\Psi_1(s)$ term by term to give $\Psi_2(1) = \sum_{j=1}^{\infty} C_1^{j-1}/j!(j+1)$; again this would appear to require truncation. Secondly, the deterministic components in Method 2 are derived from subvectors of c_t and ψ_t, and these can be shown to be in the form of linear trends by working through the algebra.

7. CONCLUDING COMMENTS

This paper has considered two approaches to deriving discrete time representations corresponding to a cointegrated system of first-order stochastic differential equations with mixed stock and flow variables. Both methods are shown to exhibit the excess differencing that was apparent in the exact discrete time model of Bergstrom (1997) corresponding to a system of mixed first- and second-order differential equations. In Method 1, which solves the system in terms of the observable vector, the source of the differencing is identified to be an equation linking the differences of the stock variables to the partially unobservable cointegrating relationships which have to be solved out of the system, while in Method 2, which solves the system in terms of the cointegrating relationships and their orthogonal counterparts, the source of the differencing is embedded within a particular matrix exponential. Furthermore, the excess differencing is shown to be due to the simultaneous presence of stock variables and flow variables, and arises because of the necessity to treat the variables differently in order to relate them to the discrete time observations. The excess differencing does not arise in systems containing only stock variables or only flow variables, in which cases the discrete time models are shown to be in the form of error correction models.

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APPENDIX A: PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. We shall first establish (8). The last n^f equations of (5) are

$$y_t^f = J_{fs} w_{t-1}^s + J_{ff} y_{t-1}^f + m_f + \gamma_f t + v_t^f. \tag{23}$$

From (3) it follows that $J_{fs} = \alpha_f F \beta_s'$ and $J_{ff} = I_{n^f} + \alpha_f F \beta_f'$, hence (23) can be written

$$\Delta y_t^f = \alpha_f F \left(\beta_s' w_{t-1}^s + \beta_f' y_{t-1}^f \right) + m_f + \gamma_f t + v_t^f. \tag{24}$$

Multiplying the first n^s equations of (6) by α_s' and rearranging we obtain

$$(\alpha_s' \alpha_s) \left(\beta_s' w_t^s + \beta_f' y_t^f \right) = \alpha_s' \left(\Delta y_t^s - g_s - b_s t - e_t^s \right).$$

Solving and lagging by one period yields

$$\beta_s' w_{t-1}^s + \beta_f' y_{t-1}^f = P_s \left(\Delta y_{t-1}^s - g_s - b_s (t-1) - e_{t-1}^s \right). \tag{25}$$

Substituting (25) into (24) results in the equation for the flow variables:

$$\Delta y_t^f = \alpha_f F P_s \Delta y_{t-1}^s + \phi_1^f + \phi_2^f t + u_t^f. \tag{26}$$

Next, differencing the first n^s equations of (4) yields

$$\Delta y_t^s = J_{ss} \Delta y_{t-1}^s + J_{sf} \Delta w_{t-1}^f + \gamma_s + \Delta \eta_t^s, \tag{27}$$

while first-differencing the last n^f equations of (6) yields

$$\Delta w_t^f = g_f + b_f t + \alpha_f \left(\beta_s' w_t^s + \beta_f' y_t^f \right) + e_t^f. \tag{28}$$

Lagging by one period and using (25) gives

$$\Delta w_{t-1}^f = g_f + b_f (t-1) + \alpha_f P_s \left(\Delta y_{t-1}^s - g_s - b_s (t-1) - e_{t-1}^s \right) + e_{t-1}^f, \tag{29}$$

which can be substituted into (27) to yield

$$\Delta y_t^s = (J_{ss} + J_{sf} \alpha_f P_s) \Delta y_{t-1}^s + \phi_1^s + \phi_2^s t + u_t^s. \tag{30}$$

Stacking (30) above (26) gives (8) as required. To derive (7), first set $t = 1$ in (4) and take the first n^s equations:

$$y_1^s = J_{ss} y^s(0) + J_{sf} y^f(0) + \phi_0^s + u_1^s. \tag{31}$$

Next, integrating (2) over the interval $(0, 1]$ gives

$$\int_0^1 y(r) dr = G y(0) + G_0 a + H_0 b + \rho_1, \tag{32}$$

the last n^f equations of which imply that

$$y_1^f = G_{fs} y^s(0) + G_{ff} y^f(0) + m_0^f + \rho_1^f. \tag{33}$$

Stacking (31) above (33) yields (7) as required. □

Proof of Theorem 2. We shall establish (20) and (21) first. From (15) and the form of e^C we find that

$$\begin{aligned} z_1(t) &= \Theta z_1(t-1) + c_{1t} + \epsilon_{1t}, & (34)\\ \Delta z_2(t) &= \Gamma z_1(t-1) + c_{2t} + \epsilon_{2t}. & (35) \end{aligned}$$

From (34) we then obtain

$$\begin{aligned} \beta_s' y_t^s &= \Theta_{ss}\beta_s' y_{t-1}^s + \Theta_{sf}\beta_f' w_{t-1}^f + c_{1t}^s + \epsilon_{1t}^s, & (36)\\ \beta_f' w_t^f &= \Theta_{fs}\beta_s' y_{t-1}^s + \Theta_{ff}\beta_f' w_{t-1}^f + c_{1t}^f + \epsilon_{1t}^f. & (37) \end{aligned}$$

Solving (36) for $\beta_f' w_{t-1}^f$ and lagging by a further period results in the pair of equations

$$\begin{aligned} \beta_f' w_{t-1}^f &= \Theta_{sf}^{-1}\left(\beta_s' y_t^s - \Theta_{ss}\beta_s' y_{t-1}^s - c_{1t}^s - \epsilon_{1t}^s\right), & (38)\\ \beta_f' w_{t-2}^f &= \Theta_{sf}^{-1}\left(\beta_s' y_{t-1}^s - \Theta_{ss}\beta_s' y_{t-2}^s - c_{1,t-1}^s - \epsilon_{1,t-1}^s\right), & (39) \end{aligned}$$

while lagging (37) by one period gives

$$\beta_f' w_{t-1}^f = \Theta_{fs}\beta_s' y_{t-2}^s + \Theta_{ff}\beta_f' w_{t-2}^f + c_{1,t-1}^f + \epsilon_{1,t-1}^f. \quad (40)$$

Substituting the right hand sides of (38) and (39) into (40) and solving for $\beta_s' y_t^s$ yields

$$\beta_s' y_t^s = K_1^s \beta_s' y_{t-1}^s - K_2^s \beta_s' y_{t-1}^s + \tau_{1t}^s + \xi_{1t}^s, \quad (41)$$

where $K_1^s = \Theta_{ss} + \Theta_{sf}\Theta_{ff}\Theta_{sf}^{-1}$. For the flows, integrating (15) over $(t-1, t]$ yields

$$\int_{t-1}^{t} z(r)dr = e^C \int_{t-2}^{t-1} z(r)dr + \psi_t + \nu_t. \quad (42)$$

From the form of e^C we have the pair of equations

$$\begin{aligned} \int_{t-1}^{t} z_1(r)dr &= \Theta \int_{t-2}^{t-1} z_1(r)dr + \psi_{1t} + \nu_{1t}, & (43)\\ \Delta \int_{t-1}^{t} z_2(r)dr &= \Gamma \int_{t-2}^{t-1} z_1(r)dr + \psi_{2t} + \nu_{2t}. & (44) \end{aligned}$$

Partitioning (43) we obtain

$$\begin{aligned} \beta_s' w_t^s &= \Theta_{ss}\beta_s' w_{t-1}^s + \Theta_{sf}\beta_f' y_{t-1}^f + \psi_{1t}^s + \nu_{1t}^s, & (45)\\ \beta_f' y_t^f &= \Theta_{fs}\beta_s' w_{t-1}^s + \Theta_{ff}\beta_f' y_{t-1}^f + \psi_{1t}^f + \nu_{1t}^f, & (46) \end{aligned}$$

which can be solved in the same way as (36) and (37) to yield

$$\beta_f' y_t^f = K_1^f \beta_f' y_{t-1}^f - K_2^f \beta_f' y_{t-1}^f + \tau_{1t}^f + \xi_{1t}^f, \quad (47)$$

where $K_1^f = \Theta_{ff} + \Theta_{fs}\Theta_{ss}\Theta_{fs}^{-1}$. Combining (41) and (47) we obtain

$$\beta' y_t = K_1^s \beta_s' y_{t-1}^s + K_1^f \beta_f' y_{t-1}^f - K_2^s \beta_s' y_{t-2}^s - K_2^f \beta_f' y_{t-2}^f + \overline{\tau}_{1t} + \overline{\xi}_{1t}. \quad (48)$$

But, from Proposition 1 in Appendix C, $K_1^s = K_2^s + I_r$ and $K_1^f = K_2^f + I_r$; making these substitutions yields (20) as required. In orthogonal directions to β, we obtain from (35)

$$\beta'_{s\perp}\Delta y^s_t = \Gamma_{ss}\beta'_s y^s_{t-1} + \Gamma_{sf}\beta'_f w^f_{t-1} + c^s_{2t} + \epsilon^s_{2t}. \tag{49}$$

Substituting (38) for $\beta'_f w^f_{t-1}$ and rearranging yields

$$\beta'_{s\perp}\Delta y^s_t = M^s_0\beta'_s y^s_t + M^s_1\beta'_s y^s_{t-1} + \tau^s_{2t} + \xi^s_{2t}, \tag{50}$$

where $M^s_0 = \Gamma_{sf}\Theta^{-1}_{sf}$ and $M^s_1 = \Gamma_{ss} - \Gamma_{sf}\Theta^{-1}_{sf}\Theta_{ss}$. Similarly, from (44) we obtain

$$\beta'_{f\perp}\Delta y^f_t = \Gamma_{fs}\beta'_s w^s_{t-1} + \Gamma_{ff}\beta'_f y^f_{t-1} + \psi^f_{2t} + \nu^f_{2t}. \tag{51}$$

Substituting for $\beta'_s w^s_{t-1}$ using an equation analogous to (38) and rearranging yields

$$\beta'_{f\perp}\Delta y^f_t = M^f_0\beta'_f y^f_t + M^f_1\beta'_f y^f_{t-1} + \tau^f_{2t} + \xi^f_{2t}, \tag{52}$$

where $M^f_0 = \Gamma_{fs}\Theta^{-1}_{fs}$ and $M^f_1 = \Gamma_{ff} - \Gamma_{fs}\Theta^{-1}_{fs}\Theta_{ff}$. Combining (50) and (52), and using (41) and (47) to substitute for $\beta'_s y^s_t$ and $\beta'_f y^f_t$ respectively, yields, after some manipulation and simplification of the matrices,

$$\beta'_{\perp}\Delta y_t = N^s_1\beta'_s y^s_{t-1} + N^f_1\beta'_f y^f_{t-1} + N^s_2\beta'_s y^s_{t-2} + N^f_2\beta'_f y^f_{t-2} + \overline{\tau}_{2t} + \overline{\xi}_{2t}, \tag{53}$$

where $N^s_2 = \Gamma_{sf}(\Theta_{fs} - \Theta_{ff}\Theta^{-1}_{sf}\Theta_{ss})$ and $N^f_2 = \Gamma_{fs}(\Theta_{sf} - \Theta_{ss}\Theta^{-1}_{fs}\Theta_{ff})$. Using Proposition 1 in Appendix C, which shows that $N^s_2 = -N^s_1$ and $N^f_2 = -N^f_1$, results in (21). Turning to (16), setting $t = 1$ in (36) gives the stock component

$$\beta'_s y^s_1 = \Theta_{ss}\beta'_s y^s(0) + \Theta_{sf}\beta'_f y^f(0) + c^s_{11} + \epsilon^s_{11}. \tag{54}$$

For the flow component, integrate (14) over $(0, 1]$:

$$\begin{aligned}\int_0^1 z(r)dr &= \int_0^1 e^{rC}dr z(0) + \int_0^1\int_0^r e^{(r-s)C}\left(\overline{a} + \overline{b}s\right)dsdr + \int_0^1\int_0^r e^{(r-s)C}\overline{\zeta}(ds)dr \\ &= \int_0^1 e^{rC}dr z(0) + \widetilde{\psi}_1 + \widetilde{\nu}_1,\end{aligned} \tag{55}$$

Because

$$\int_0^1 e^{rC}dr = \begin{bmatrix}\Psi_0 & 0\\ \widetilde{\Gamma} & I\end{bmatrix},$$

the equations involving z_1 are

$$\int_0^1 z_1(r)dr = \Psi_0 z_1(0) + \widetilde{\psi}_{11} + \widetilde{\nu}_{11}. \tag{56}$$

Picking out the flow components:

$$\beta'_f y^y_1 = \Psi^{fs}_0 y^s(0) + \Psi^{ff}_0 y^f(0) + \widetilde{\psi}^f_{11} + \widetilde{\nu}^f_{11}. \tag{57}$$

Combining (54) and (57) yields (16). In orthogonal directions, setting $t = 1$ in (49) yields

$$\beta'_{s\perp}\Delta y^s_1 = \Gamma_{ss}\beta'_s y^s(0) + \Gamma_{sf}\beta'_f y^f(0) + c^s_{21} + \epsilon^s_{21}. \tag{58}$$

From (55), the equations for z_2 are

$$\int_0^1 z_2(r)dr = \widetilde{\Gamma} z_1(0) + z_2(0) + \widetilde{\psi}_{21} + \widetilde{\nu}_{21}, \tag{59}$$

the flow component of which is

$$\beta'_{f\perp} y_1^f = \widetilde{\Gamma}_{fs} \beta'_s y^s(0) + \widetilde{\Gamma}_{ff} \beta'_f y^f(0) + \beta'_{f\perp} y^f(0) + \widetilde{\psi}_{21}^f + \widetilde{\nu}_{21}^f. \tag{60}$$

Combining (58) and (60) yields (17) as required. Turning to (18), setting $t = 2$ in (36) yields

$$\beta'_s y_2^s = \Theta_{ss} \beta'_s y_1^s + \Theta_{sf} \beta'_f w_1^f + c_{12}^s + \epsilon_{12}^s, \tag{61}$$

while putting $t = 1$ in (37) gives

$$\beta'_f w_1^f = \Theta_{fs} \beta'_s y^s(0) + \Theta_{ff} \beta'_f y^f(0) + c_{11}^f + \epsilon_{11}^f. \tag{62}$$

Substituting (62) into (61) yields the stock component of (18):

$$\beta'_s y_2^s = \Theta_{ss} \beta'_s y_1^s + \Theta_{sf} \Theta_{fs} \beta'_s y^s(0) + \Theta_{sf} \Theta_{ff} \beta'_f y^f(0) + c_{12}^s + \Theta_{sf} c_{11}^f + \epsilon_{12}^s + \Theta_{sf} \epsilon_{11}^f. \tag{63}$$

For the flows, putting $t = 2$ in (46) yields

$$\beta'_f y_2^f = \Theta_{fs} \beta'_s w_1^s + \Theta_{ff} \beta'_f y_1^f + \psi_{12}^f + \nu_{12}^f, \tag{64}$$

while setting $t = 1$ in (45) yields

$$\beta'_s w_1^s = \Theta_{ss} \beta'_s y^s(0) + \Theta_{sf} \beta'_f y^f(0) + \psi_{11}^s + \nu_{11}^s. \tag{65}$$

Substituting (65) into (64) gives

$$\beta'_f y_2^f = \Theta_{ff} \beta'_f y_1^f + \Theta_{fs} \Theta_{ss} \beta'_s y^s(0) + \Theta_{fs} \Theta_{sf} \beta'_f y^f(0) + \psi_{12}^f + \Theta_{fs} \psi_{11}^s + \nu_{12}^f + \Theta_{fs} \nu_{11}^s. \tag{66}$$

Combining (63) and (66) yields

$$\beta' y_2 = \Theta_{ss} \beta'_s y_1^s + \Theta_{ff} \beta'_f y_1^f + K_{20}^s \beta'_s y^s(0) + K_{20}^f \beta'_f y^f(0) + \overline{\tau}_{12} + \overline{\xi}_{12},$$

where $K_{20}^s = \Theta_{sf} \Theta_{fs} + \Theta_{fs} \Theta_{ss}$ and $K_{20}^f = \Theta_{sf} \Theta_{ff} + \Theta_{fs} \Theta_{sf}$. But, using the fact that $\Theta_{fs} = \Theta_{ff} - I_r$ and $\Theta_{sf} = \Theta_{ss} - I_r$ and making these substitutions, we can show that $K_{20}^s = \overline{K}_{20} - \Theta_{ss}$ and $K_{20}^f = \overline{K}_{20} - \Theta_{ff}$, yielding (18) as required. Finally, for the orthogonal components, putting $t = 2$ in (49) gives

$$\beta'_{s\perp} \Delta y_2^s = \Gamma_{ss} \beta'_s y_1^s + \Gamma_{sf} \beta'_f w_1^f + c_{22}^s + \epsilon_{22}^s. \tag{67}$$

Substituting (62) into (67) gives

$$\beta'_{s\perp} \Delta y_2^s = \Gamma_{ss} \beta'_s y_1^s + \Gamma_{sf} \Theta_{fs} \beta'_s y^s(0) + \Gamma_{sf} \Theta_{ff} \beta'_f y^f(0) + c_{22}^s + \Gamma_{sf} c_{11}^f + \epsilon_{22}^s + \Gamma_{sf} \epsilon_{11}^f. \tag{68}$$

Putting $t = 2$ in (51):

$$\beta'_{f\perp} \Delta y_2^f = \Gamma_{fs} \beta'_s w_1^s + \Gamma_{ff} \beta'_f y_1^f + \psi_{22}^f + \nu_{22}^f. \tag{69}$$

Substituting (65) into (69) yields

$$\beta_{f\perp}'\Delta y_2^f = \Gamma_{ff}\beta_f' y_1^f + \Gamma_{fs}\Theta_{ss}\beta_s' y^s(0) + \Gamma_{fs}\Theta_{sf}\beta_f' y^f(0) + \psi_{22}^f + \Gamma_{fs}\psi_{11}^s + \nu_{22}^f + \Gamma_{fs}\nu_{11}^s \quad (70)$$

which, combined with (68), yields

$$\beta_\perp'\Delta y_2 = \Gamma_{ss}\beta_s' y_1^s + \Gamma_{ff}\beta_f' y_1^f + N_{20}^s\beta_s' y^s(0) + N_{20}^f\beta_f' y^f(0) + \overline{\tau}_{22} + \overline{\xi}_{22},$$

where $N_{20}^s = \Gamma_{sf}\Theta_{fs} + \Gamma_{fs}\Theta_{ss}$ and $N_{20}^f = \Gamma_{sf}\Theta_{ff} + \Gamma_{fs}\Theta_{sf}$. But, from Proposition 1 in Appendix C, $\Gamma_{sf} = \Gamma_{ss}$ and $\Gamma_{fs} = \Gamma_{ff}$, so that $N_{20}^s = \overline{N}_{20} - \Gamma_{ss}$ and $N_{20}^f = \overline{N}_{20} - \Gamma_f$, which results in (19) as required. □

APPENDIX B: PROOFS OF THEOREMS 3, 4, AND 5

Proof of Theorem 3. First note that $\eta_t = \int_{t-1}^t F_0(t-s)\zeta(ds)$. We can also write

$$\begin{aligned} v_t &= \int_{t-1}^t \int_{r-1}^r e^{(r-s)A}\zeta(ds)dr \\ &= \int_s^t \int_{t-1}^t e^{(r-s)A}\zeta(ds)dr + \int_{t-1}^{s+1}\int_{t-2}^{t-1} e^{(r-s)A}\zeta(ds)dr \\ &= \int_{t-1}^t F_1(t-s)\zeta(ds) + \int_{t-2}^{t-1} F_2(t-1-s)\zeta(ds), \end{aligned} \quad (71)$$

where $F_1(t-s) = \int_s^t e^{(r-s)A}dr$ and $F_2(t-1-s) = \int_{t-1}^{s+1} e^{(r-s)A}dr$. The expressions for $F_1(s)$ and $F_2(s)$ are obtained by making the appropriate substitutions. In a similar way we obtain

$$\rho_1 = \int_0^1\int_0^r e^{(r-s)A}\zeta(ds)dr = \int_s^1\int_0^1 e^{(r-s)A}\zeta(ds)dr = \int_0^1 F_1(1-s)\zeta(ds). \quad (72)$$

From the definitions we can write $u_t = S_1\eta_t - S_1\eta_{t-1} + S_2 v_t + S_3 e_{t-1}$ and $u_1 = S_1\eta_1 + S_2\rho_1$. Substituting for η_t, v_t and e_t we obtain

$$u_1 = \int_0^1 J_0(1-s)\zeta(ds), \quad u_t = \int_{t-1}^t J_0(t-s)\zeta(ds) + \int_{t-2}^{t-1} J_1(t-1-s)\zeta(ds), \quad t = 2,\ldots,T.$$

The autocovariances are derived from these latter expressions. □

Proof of Theorem 4. First note that $\epsilon_t = \int_{t-1}^t R_0(t-s)\zeta(ds)$ while, proceeding as in the proof of Theorem 3,

$$\begin{aligned} \nu_t &= \int_{t-1}^t \int_{r-1}^r e^{(r-s)C}\overline{\zeta}(ds)dr \\ &= \int_s^t \int_{t-1}^t e^{(r-s)C}\overline{\zeta}(ds)dr + \int_{t-1}^{s+1}\int_{t-2}^{t-1} e^{(r-s)C}\overline{\zeta}(ds)dr \\ &= \int_{t-1}^t R_1(t-s)\zeta(ds) + \int_{t-2}^{t-1} R_2(t-1-s)\zeta(ds), \end{aligned} \quad (73)$$

recalling that $\overline{\zeta}(dt) = K'\zeta(dt)$. Similarly it can be shown that $\nu_1 = \int_0^1 R_1(1-s)\overline{\zeta}(ds)$. From the definitions of $\overline{\xi}_{1t}$ and $\overline{\xi}_{2t}$ it follows that $\overline{\xi}_t$ has the representation $\overline{\xi}_t = S_4\epsilon_t + S_5\epsilon_{t-1} + S_6\nu_t + S_7\nu_{t-1}$ while $\overline{\xi}_1 = S_4\epsilon_1 + S_6\nu_1$ and $\overline{\xi}_2 = S_4\epsilon_2 + S_8\epsilon_1 + S_6\nu_2 + S_9\nu_1$. Substituting the expressions for ϵ_t and ν_t in terms of the integrals with respect to $\zeta(dt)$ we obtain the representations

$$\bar{\xi}_1 = \int_0^1 W_0(1-s)\zeta(ds), \quad \bar{\xi}_2 = \int_1^2 W_0(2-s)\zeta(ds) + \int_0^1 W_3(1-s)\zeta(ds),$$

$$\bar{\xi}_t = \int_{t-1}^t W_0(t-s)\zeta(ds) + \int_{t-2}^{t-1} W_1(t-1-s)\zeta(ds) + \int_{t-3}^{t-2} W_2(t-2-s)\zeta(ds)$$

for $t = 3, \ldots, T$, from which the autocovariances are straightforwardly derived. □

Proof of Theorem 5. From (3) we can write $e^{sA} = \Pi_1 + \Pi_2 e^{sB}\beta'$, from which

$$G = \int_0^1 (\Pi_1 + \Pi_2 e^{sB}\beta')ds = \Pi_1 + \Pi_2 \int_0^1 e^{sB} ds\beta'.$$

Noting that $\int_0^1 e^{sB}ds = B^{-1}(e^B - I_r) = F$ yields the required expression. The matrix H also involves e^{sA} and so, using the representation above,

$$H = \Pi_1 \int_0^1 sds + \Pi_2 \int_0^1 se^{sB}ds\beta'.$$

Now $\int_0^1 se^{sB}ds = B^{-1}[se^{sB}]_0^1 - B^{-2}[e^{sB}]_0^1 = B^{-1}e^B - B^{-2}(e^B - I_r) = \Pi_3$, resulting in the required expression. Turning to G_0, making the change of varible to $q = r - s$ we obtain $G_0 = \int_0^1[\int_0^r e^{qA}dq]dr$. Now, using previous arguments, $\int_0^r e^{qA}dq = \Pi_1 r + \Pi_2 B^{-1}(e^{rB} - I_r)\beta'$, and integrating again with respect to r yields the stated expression. For H_0, we find that

$$H_0 = \int_0^1 r\left[\int_0^r e^{qA}dq\right]dr = \Pi_1 \int_0^1 r^2 dr + \Pi_2 B^{-1}\int_0^1 re^{rB}dr\beta' - \Pi_2 B^{-1}\beta' \int_0^1 rdr$$

which results in the required expression. The expansions for $J_0(s)$ and $J_1(s)$ arise from noting that $F_0(s) = \Pi_1 + \Pi_2 e^{sB}\beta'$, $F_1(s) = \Pi_1 s + \Pi_2 B^{-1}(e^{sB} - I_r)\beta'$ and $F_2(s) = \Pi_1(1 - s) + \Pi_2 B^{-1}(e^B - e^{sB})\beta'$, the latter two of which are obtained by carrying out the integration as above. Substituting these into the expressions for $J_0(s)$ and $J_1(s)$ yields the relevant matrices J_{ik} ($i = 0, 1; k = 0, 1, 2$) which can then be plugged into the autocovariances and integrated term by term. □

APPENDIX C: SUPPLEMENTARY RESULTS

Proof of (3). From the definition of A and e^{tA} we obtain $e^{tA} = I_n + \sum_{j=1}^{\infty} t^j(\alpha\beta')^j/j!$. But $(\alpha\beta')^j = \alpha(\beta'\alpha)^{j-1}\beta'$ and so, defining $B = \beta'\alpha$,

$$e^{tA} = I_n + \alpha \sum_{j=1}^{\infty} \frac{t^j}{j!} B^{j-1}\beta' = I_n + \alpha B^{-1} \sum_{j=1}^{\infty} \frac{t^j}{j!} B^j \beta',$$

thereby yielding (3). □

PROPOSITION 1. *Let Θ and Γ be defined as in section 4, K_2^s, let K_2^f, N_1^s and N_1^f be defined as in Theorem 2, and let K_1^s, K_1^f, N_2^s and N_2^f be defined as in the Proof of Theorem 2. Then:*

(i) $K_1^s = K_2^s + I_r$ *and* $K_1^f = K_2^f + I_r$*;*

(ii) $\Gamma_{sf} = \Gamma_{ss}$ *and* $\Gamma_{fs} = \Gamma_{ff}$*;*

(iii) $N_2^s = -N_1^s$ *and* $N_2^f = -N_1^f$*.*

Proof.

(i) Because $\Theta_{ss} = \Theta_{sf} + I_r$ we find that $K_2^s = \Theta_{sf}\Theta_{ff} + \Theta_{sf}\Theta_{ff}\Theta_{sf}^{-1} - \Theta_{sf}\Theta_{fs}$. But $\Theta_{ff} = \Theta_{fs} + I_r$ implying $\Theta_{sf}\Theta_{ff} - \Theta_{sf}\Theta_{fs} = \Theta_{sf}$. Hence

$$K_2^s = \Theta_{sf} + \Theta_{sf}\Theta_{ff}\Theta_{sf}^{-1} = \Theta_{ss} - I_r + \Theta_{sf}\Theta_{ff}\Theta_{sf}^{-1} = K_1^s - I_r,$$

as required. The proof of $K_1^f = K_2^f + I_r$ follows in the same way with appropriate modifications.

(ii) Recall that $\Gamma = C_2 Q$, where $Q = \sum_{j=1}^{\infty} C_1^{j-1}/j!$. From the form of C_1 and C_2, and partitioning Q conformably, we obtain

$$\Gamma = \begin{pmatrix} C_{21} & C_{21} \\ C_{22} & C_{22} \end{pmatrix} \begin{pmatrix} Q_{ss} & Q_{ss} - I_r \\ Q_{ff} - I_r & Q_{ff} \end{pmatrix} = \begin{pmatrix} C_{21}\overline{Q} & C_{21}\overline{Q} \\ C_{22}\overline{Q} & C_{22}\overline{Q} \end{pmatrix},$$

where $\overline{Q} = Q_{ss} + Q_{ff} - I_r$, thereby establishing the claim.

(iii) We begin with the substitutions $\Theta_{ss} = \Theta_{sf} + I_r$ and $\Theta_{ff} = \Theta_{fs} + I_r$:

$$\begin{aligned} N_2^s &= \Gamma_{sf}\Theta_{fs} - \Gamma_{sf}(\Theta_{fs} + I_r)\Theta_{sf}^{-1}(\Theta_{sf} + I_r) \\ &= \Gamma_{sf}\Theta_{fs} - \Gamma_{sf}(\Theta_{fs} + I_r) - \Gamma_{sf}(\Theta_{fs} + I_r)\Theta_{sf}^{-1} \\ &= -\Gamma_{sf} - \Gamma_{sf}\Theta_{ff}\Theta_{sf}^{-1}, \end{aligned}$$

and hence $N_2^s = -N_1^s$ using part (ii) for Γ_{sf}. That $N_2^f = -N_1^f$ is proved in an identical way with the appropriate modifications. □