

COMPUTABILITY AND EVOLUTIONARY COMPLEXITY: MARKETS AS COMPLEX ADAPTIVE SYSTEMS (CAS)*

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Few will argue that the epi-phenomena of biological systems and socio-economic systems are anything but complex. The purpose of this Feature is to examine critically and contribute to the burgeoning multi-disciplinary literature on markets as complex adaptive systems (CAS). The new sciences of complexity, the principles of self-organisation and emergence along with the methods of evolutionary computation and artificially intelligent agent models have been developed in a multi-disciplinary fashion. The cognoscenti here consider that complex systems whether natural or artificial, physical, biological or socio-economic can be characterised by a unifying set of principles. Further, it is held that these principles mark a paradigm shift from earlier ways of viewing such phenomenon.

The articles in this Feature aim to provide detailed insights and examples of both the challenges and the prospects for economics that are offered by the new methods of the complexity sciences. The applicability or not of the optimisation framework of conventional economics depends on the domain of the problem and in particular the modern theories behind non-computability are outlined to explain why adaptive or emergent methods of computation and agent-based simulations are needed to understand and model a whole swathe of economic phenomena that are anomalies in mainstream economics. These include innovation, competitive co-evolution, persistent heterogeneity, increasing returns, the error driven processes behind market equilibria, herding, crashes and extreme events such as in the business cycle or in stock markets. Such a list is, of course, by no means exhaustive and the Feature is selective in its coverage.

The overview also explains how the specially commissioned papers of Robert Axtell, Arthur Robson and Steven Durlauf are respectively exemplars of the three main aspects of markets as CAS:

- (i) the formal complexity issues of market equilibria which relate to the infeasibility of their algorithmic implementation,
- (ii) the dynamic behind the growth of adaptive novelty in systems, and
- (iii) the empirical and testable manifestations of CAS in economic phenomena.

The earliest precursor to modern complex systems theory resides in the first of these precepts from the classical eighteenth-century political economy of the Scottish Enlightenment. It has been held that order in market systems is

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spontaneous or emergent: it is the result of 'human action and not the execution of human design'. This early observation, well known also from the Adam Smith metaphor of the invisible hand, premises a disjunction between system wide outcomes and the design capabilities of individuals at a micro level and the distinct absence of an external organising force. It has been claimed that not just market equilibria but many diverse institutions in society including language, division of labour, civil society and monetary exchange are unintended consequences of individuals' actions rather than those borne of rational calculation and design. This anti-creationist classical thesis in economics which marks the provenance of modern evolutionary thought is also supposed to have predated and influenced Darwin, Hodgson (1993). The major challenge implied by the anti-creationism in evolution, the so called *contra* 'argument from design' pertains to the most remarkable of macro-evolutionary trends vis-à-vis the emergence of new forms within more complex organisms or systems.

Section 1 reviews the multi-disciplinary and computational legacy of modern CAS theory and explains why the *sine qua non* of a CAS is found to be its capacity to produce novelty or 'surprises'. It will be argued there that it was not until well into the twentieth century that two epochal developments in the foundations of mathematics and advances in computer technology permitted definitive formulations of CAS in terms of what cannot be formally and practically computed (NP-hard problems) and hence needs to emerge or self-organise. The first of these is the Gödel-Turing-Post¹ results on incompleteness and algorithmically unsolvable problems, where, for the first time, the logical impossibility limits to formalistic calculation or deductive methods were established. In the absence of these limits on computation there is in principle no reason why all observed patterns and new forms cannot be achieved by central command. Indeed, Goldberg (1995) maintains that without 'a heavy dose of mechanism' based on these foundational advances on what can and cannot be calculated, it is not possible to explain novelty production which is considered to be the hallmark of CAS.² Mathematically, this has been found to be of a tall order: novelty production requires the formal conditions involving hostile agents with Turing machine-type capabilities that produce computationally undecidable dynamics. Such non-computable complex dynamics as well as NP-hard problems which are practically non-computable can only self-organise using adaptive or emergent processes.³ Novelty production and self-organisation, the former being a subset of the latter, take myriad different forms and many aspects of both are still not fully understood.

¹ The seminal papers here are Gödel (1931), Turing (1936) and Post (1944).

² Goldberg (1995) claims that the mystery shrouding innovation can be dispelled '... by a heavy dose of *mechanism*. Many of the difficulties in the social sciences comes from a lack of a *computational theory* of actor innovation ... population oriented systems are dominated by what economists call the law of unintended consequences (which is itself largely the result of the innovative capability of the actors)' (p. 28).

³ The formation of soap film along some least-path configuration is classified as emergent computation as it effectively 'solves' an NP-hard problem akin to that of the Travelling Salesman. The efficient form of the soap film emerges from the parallel action of innumerable local forces of surface tension acting in unison (Ball, 1999).

The three major natural exponents of novelty-producing CAS are evolutionary biology, immune systems and innovation-based structure changing growth of capitalist systems, dubbed creative destruction by Schumpeter (1950). The perspectives on self-organised complexity arising from the different sciences involved here, viz. Computer Science/Mathematics, Physics, Biology and Economics have much in common. However, as discussed in Sections 1.2 and 1.3, different postulates exist on what are regarded to be the testable and observable aspects of CAS depending on whether or not the emergence of new forms or the capacity for variety is endogenous to the process of self-organisation.

The second significant methodological development relating to the above is the Holland-Bak-Arthur use of computer-based artificial environments to simulate dynamics from large numbers of interacting agents with varying levels of computational and adaptive intelligence to give evidence for the otherwise elusive phenomenon of emergence and self-organisation. In all variants of complex systems theory it is held that macroscopic properties cannot be formally or analytically deduced from the properties of its parts. Methodologically, it is precisely this that distinguishes the sciences of complex systems from the bulk of traditional science which relies on deductive formalistic and analytical methods.

Though many of these advances in the methodology of science have bypassed mainstream economics (Krugman, 1996), the contributions of some economists to CAS theory have been substantial. The seminal work of Schelling (1978) is one of the earliest examples of the use of computer simulation to demonstrate how simple micro behavioural rules result in a self-organised macro outcome, an undesirable one of racial segregation, which could not have been deduced from the initial rules.⁴ More recently, a number of economists and physicists have got involved in the new computational agent based modelling in economics and they have respectively been called ACEs (Agent-based Computational Economists) (Testfason, 1998) and econo-physicists.

Section 2 briefly discusses the computational and complexity issues relating to price formation and market equilibria. A major part of self-organisation in markets revolves around learning rational expectations equilibria or fixed point mappings of strongly self-referential system wide properties. In Section 2.1, I discuss the Feature paper by Robert Axtell on how decentralised arrangements for exchange overcome the NP-hard problem in Walrasian models and permit feasible real time polynomial complexity in price determination. In Section 2.2 Arthur's (1994) famous El Farol or minority game with its contrarian structure is given to illustrate the fundamental problem of inductive inference in prices in pure speculative markets which militates against homogenous rational expectations.

Section 3 outlines some recent seminal developments on the issues relating to the emergence of new forms and the growth of complexity. Much of this discussion can be subsumed under the Red Queen principle on competitive co-evolution that was first discussed in the economics literature by Robson (2002) and in this Feature. In

⁴ In Schelling's racial segregation model, households move residence because of a weak preference for a neighbourhood that has at least one third of those adjoining them to be of the same colour or race. Over time this results in very segregated neighbourhoods though everyone is initially placed at random and nobody is particularly intolerant.

Section 3.2 some economic examples of the Red Queen such as competitive product innovation, the Lucas (1972) postulates on regulatory arbitrage and on the strategic use of 'surprises' are given. Section 3.3 highlights the significance of oppositional structures or of 'parasites' in the emergence of new forms which becomes a means by which systems escape from entrapment at local optima to global optima. The necessity of 'parasites' or hostile agents is shown to be pertinent both in experimental artificial environments as well as in the formal logic of Gödel (1931).

Section 4 covers some ground on the relationship between empirical economics in terms of testable hypotheses of CAS theory. In Section 4.1, a brief overview of Durlauf's contribution shows that the two main traditional methods of theory validation with varying degrees of rigour – the historical or case study method and econometric analysis – have for the most part failed to give evidence that can be specifically adduced to CAS. In other words, as other explanations can fit the bill or the power of econometric tests is weak, identification problems loom large. The latter is true also for agent-based models. In Sections 4.2 and 4.3, the Red Queen effects and a lack of structural invariance and failure of meta (econometric) models to identify strategically induced innovation-based structural change are shown to be important additions to the so called stylised CAS facts to do with lock ins, path dependence, network effects, nonlinearities from thresholds and self-referential calculations, power law distributions, long memory and fat tails. This is followed by a brief concluding section.

1. Modern Complex Systems Theory

1.1. *The Multi-Disciplinary and Computational Legacy*

Pioneering multi disciplinary work in this area, without being exhaustive, has included that of computer scientists and mathematicians (von Neumann, 1970; Wolfram, 1984; Penrose, 1988; Holland, 1975, 1992, 1998; Koza, 1992; Goldberg, 1989; Mandelbrot, 1982). Physicists (Nicolis and Prigogine, 1977, 1989; Bak *et al.*, 1988; Bak, 1996; Anderson *et al.*, 1988; Langton *et al.*, 1992); biologists (Kauffman, 1993; Green, 1994); economists and political scientists (Day and Chen, 1993; Axelrod, 1984, 1987; Dosi and Nelson, 1994; Epstein and Axtell, 1996; Krugman, 1996; Arthur *et al.*, 1997; Albin, 1998; Velupillai, 2000). A number of these have been associated with the Santa Fe Institute.

It is the work of John von Neumann in the 1940s on self-reproducing machines as models for biological systems and self-organised complexity⁵ which provides a landmark transformation of dynamic systems theory based on motion, force and energy to the capabilities and constraints of information processors modelled as computing machines. The von Neumann models based on cellular automata⁶ have laid the ground rules of modern complex systems theory regarding:

⁵ Mirowski (2000) discusses how radical a shift this has been for the methodology of science and also from the perspective of von Neumann's earlier work with Oscar Morgenstern on the *Theory of Games and Economic Behaviour*.

⁶ Cellular automata were developed by von Neumann and Stanislaw Ulam to represent biological systems and for the purpose of modelling biological self-reproduction.

- (A) the use of large ensembles of micro level computational entities or automata following simple rules of local interaction and connectivity,
- (B) the capacity of these computational entities to self-reproduce and also to produce automata of greater complexity than themselves, and
- (C) use of the principles of computing machines to explain diverse system-wide or global dynamics.

The significance of the von Neumann computational legacy of CAS is that it covers all substrata, ranging from the bio-chemical to the artificial, in which effective procedures or computation reside. By the Church-Turing thesis (Cutland, 1980) the intuitive notion of effective procedures or an algorithm can be identified with the class of general recursive functions and represent finitely encodable programs implemented in a number of equivalent ways referred to as automata or mechanism. The best known among these idealisations of mechanism is the Turing machine and no mechanism can exceed the computational powers of Turing machines. Such a definition of mechanism or formalistic calculation is necessary before complexity measures of the disjunction between the microscopic elements of the system and their macroscopic properties can be ascertained and also on what constitutes an innovation or surprise in the system.⁷ Further, there are compelling reasons why the powerful agenda of Simon (1956, 1976) on the procedural lacunae of rationality to further our understanding of observable boundedly rational behaviour, should be securely based on modern computability theory on the limits and efficacy of effective procedures.

In keeping with (A) above, as observed by Arthur (1991), the units of modern adaptive agent models are ‘parameterised decision algorithms’ or units whose behaviour is brought about by finitely encodable algorithms. Indeed, as noted by Langton (1992) physical dynamic systems ‘are bound by the same *in principle* limitations as computing devices’ (p. 82). These limitative results of computing devices are generically referred to as the halting problem. Church’s Theorem and in particular the Gödel (1931) First Incompleteness Theorem show how Turing machines themselves can produce encoded objects (viz. by mechanising the exit route in Georg Cantor’s famous diagonal method) that cannot be enumerated by any machine. Such objects are innovations in the system and technically do not belong to recursively or algorithmically enumerable sets on which Turing machines halt. With regard to this Mirowski (2000) has correctly asserted that mathematicians ‘finally have blazed the trail to a formalised logical theory of evolution’ (p.141). In other words, dynamic system outcomes produced by algorithmic agents need not be computable and fail to be systematically identified by codifiable meta models. This is referred to as undecidable dynamics. Gödel’s Second Incompleteness Result shows that it is precisely when systems isomorphic

⁷ In a recently edited volume by Colander (2000) on the ‘Complexity Vision’ in the teaching of economics, the methodological advances made possible for the analysis of complexity by computer simulations are highlighted and so are a number of stylised CAS facts (see Section 4 of this article). However, there has been little or no mention there of the advances in formalistic mathematics that has led to the capacity for novelty production rather than nonlinearity or chaos as being the *sine qua non* of CAS. Consequently, in Colander (2000) there is simply little scope to probe into issues on how systems become and remain complex (see, Section 3 and Robson in this Feature).

to number theory are consistent that internal consistency, which is a strongly self-referential system wide property often regarded as the hallmark of rational order, cannot be established by an algorithmic decision procedure. Gödel (1931) axiomatically derived the undecidable proposition, the encoding of which represents the diophantine equation which has no algorithmic solution.⁸ This class well known as Hilbert's Tenth problem has the highest degree of algorithmic unsolvability. Penrose (1988) was amongst the first to identify so called non-computable patterns or tiling problems, that nevertheless emerge from the execution of simple rules, with the Gödel incompleteness result.

Methods of search and inductive inference in non-computable domains of problems which defy algorithmic or deductive inference has progressed substantially with methods of evolutionary/adaptive computation such as artificial neural networks (ANN) (Linster, 1992) and in the form of genetic algorithms (GA) and genetic programs (GP) pioneered, respectively, by Holland (1975) and Koza (1992). The fundamental issue with search procedures for optimal solutions is that the domain of search of all possible procedures which are fully defined, viz. the set of total recursive functions, is not recursively enumerable. This set which is uncountably infinite has no algorithmic decision procedure and hence trial and error outcomes of inductive search may be far from a global optimum. This implies that the optimisation framework of conventional economics is based on an artefact of restricting choice to a given and fixed set and hence at best obtains only local optima. Typically, algorithmically unsolvable problems fail to be categorical⁹ in that they have no unique decision/solution procedure. Instead, a multiplicity of heterogeneous decision rules have to be considered in parallel and distributed fashion. This is the hallmark of methods of many natural and artificial evolutionary computation such as GAs, GPs and ANNs. Holland (1975) pioneered the Darwin/Fisher principle for the selection of decision rules within the context of GAs. A decision rule is selected to grow/replicate at a rate which is in proportion to its fitness relative to the average fitness of the fixed population of decision rules. Hence, the term replicator dynamics has been given to this method of selection (Hofbauer and Sigmund, 1998, p. 67) and is thought to suit best short-term evolutionary dynamics with fixed number of types in the population. Note that GAs and GPs can introduce novel decision rules by cross over and mutation operators. In general, the problem of selection when global credit assignment mechanisms on individual performance are not in place or how selective processes evolve from local interactions falls under the major theme of self-organisation in CAS. The significant factors in the long-run macro evolutionary trends, viz. endogenous enhancement of performance/fitness or enlarging choice domains by innovations, are increasingly being seen as intelligent strategic adaptive responses in coevolving heterogeneous populations often referred to as the Red Queen principle. Artificial

⁸ Diophantine equations are polynomial equations with integer solutions. The irreducible nature of the computation here is that short of letting the system run its course there is no *a priori* systematic way to determine the solution to the problem.

⁹ At the turn of the last century, positive proof of categoricity along with completeness and consistency of formal systems, was stipulated as desiderata of formal systems by the mathematician David Hilbert. Gödel (1931) overturned this as a logical impossibility.

life multi agent simulations pioneered at the Santa Fe Institute and elsewhere aid in a deeper understanding of the nexus between the non-anticipating nature of global outcomes and individual agent programs or rationality.

Table 1, first developed in Markose (2002) and embellished further here, shows how the new evolutionary computational methods on inductive inference and emergent computation relate to the more traditional formalistic/deductive methods. What should be noted from Table 1 is that the axiomatically proven impossibility results on computability (Column II) delineate the classes of problems that mathematically fail to be amenable to deductive methods and why evolutionary adaptive methods have to be resorted to. Further, one may contrast the implications for economics given at the bottom of Columns I, II and III of deducibility and incompleteness, respectively.

It is a major premise of this overview that scant attention paid by economists to the epochal non-computability issues for their science has been a stumbling block to a deeper understanding of the relevance of evolutionary methods and the principles of self-organisation given in Column III, Table 1. Hayek (1952, 1967, 1982) was one of the first economists who explicitly espoused the Gödelian formalist incompleteness and impossibility limits on calculation which he referred to as the limits of constructivist reason. This led Hayek to the necessity of experientially driven adaptive solutions with the abiding premise of his large oeuvre of work being that market institutions which co-evolved with human reason enable us to solve problems which are impossible to do so by direct rational calculation. Using different formulations of the problem, Lewis (1987) and Spear (1989) seminaly showed the generic non-computability of fixed point mappings that represent equilibria in markets. Axtell's contribution to this Feature, and pioneering work by Arifovic (1994) on evolutionary learning of rational expectations and inductive methods for equilibrium selection (see, survey in Arifovic, 2003) arise because of non-computability issues or because of real time computational intractability of fixed point mappings. Velupillai (2000) derived the diophantine complexity class of parallel distributed computing that arises in a game where cooperation and competitive innovation can arise. The non-computability of fixed points with contrarian or hostile players, the mainstay of Gödelian logic and that of the halting problem will also be shown to be pertinent to the very large area pioneered by Brian Arthur using the minority game model to illustrate the nature of self-organisation in stock market environments. As we will see, the Binmore (1987) use of the Gödelian logic and Turing machine agents in a game is shown by Markose (2001*b*, 2002, Section 3) to be of fundamental importance in resolving one of the mysteries surrounding CAS on the emergence of new forms. Thus, while Velupillai (2000) succinctly points out how problems of non-computability naturally reside close at hand for cognitive aspects of choice, following Binmore (1987) this overview, in Section 3, underscores theoretical reasons that market agents can be assumed to have full computational capabilities of Turing machines and why in no small measure the non-computable environment that necessitates the inductive methods of inference and problem solving in Table 1 Column III are endogenously generated by the activity of such agents.

In the case of the choice problem, even a simple evaluation of the truth value of 'X or else Y' type proposition which is necessary for categorisation of objects of

Table 1

I. Formalist/ Deductive Methods and Related Dynamical Systems	II. The 'New Logic': Mathematics of Incompleteness and NP-hard Problems	III. Inductive Methods and Self Organising Dynamics
# Axiomatic Proof Theory	# Gödel (1931): Self-reference, Undecidability and Incompleteness	# Theory of Emergence and Self-Organising Complex Systems (Local interaction models with spatial topology of networks: <i>with</i> and <i>without</i> endogenous production of adaptive novelty)
# Model Theory	# Church-Turing-Post: Algorithmic Unsolvability; Halting Problems # Wolfram-Langton Complexity class with Type IV dynamics and phase transition: 'life at the edge of chaos'. <i>Computability Methods</i>	# Non-computable Penrose tiling and pattern generation # Coevolving multi-species and Red Queen Effects
<i>Formalistic Methods</i>	* Recursion Function / Computability Theory * Algorithmic and Stochastic Complexity Theory	<i>Methods of Adaptive Computing</i>
* Predicate and Propositional Calculus * Classical methods of optimisation		* Cellular Automata * Classifier Systems
* Classical Probability and Econometric Models		* Genetic Algorithms and Genetic Programs * Neural Networks * Numerical multi-agent simulations
• Perfect competition; existence results in general equilibrium models and static welfare analysis	• Hayek (1967, 1982) on limits of constructivist reason; Hayek (1952, 1982) on cognitive incompleteness	• Irregular innovation based structure changing dynamics in capitalist growth: Type IV undecidable dynamics
• Command Economy	• Lewis (1985, 1987) on algorithmic unsolvability of general equilibrium prices	• Stock market crashes and non Gaussian asset returns with fat tails
• Evolutionary Game Theory and diffusion dynamics with no new forms	• Spear (1989) on generic non-computability of Rational Expectations fixed points	• Red Queen dynamics with rivalrous arms race in product innovation, regulatory arbitrage etc.
• Lognormal stock market models and no trade results	• Velupillai (2000) on Diophantine complexity of global encoding of parallel distributed computation	• Critical heterogeneity
• Homogenous rational expectations	• Binmore (1987)/Markose (2001/2, 2002, 2004)	• Power laws, fractals and scaling
• Uniqueness and homogeneity with Type I (limit points), Type II (limit cycle) dynamics	non-computability of fixed points with contrarian agents and strategic innovation	• Information/technology diffusion in socio-economic networks with path dependence, increasing returns to scale
• Market Completeness	• Axtell (in this Feature) NP-hardness of Walrasian Equilibria	• Evolved norms and institutions for self-organisation: computational efficiency versus maladaptive entrainment
		• Market Incompleteness

Implications for Economic Models

choice, involves calculations of the neuronal network that are nonlinear, of over dimension three, along with the assignment of integer or rational valued weights as in the solution of a diophantine equation, Velupillai (2000). As no uniform solution procedures exist for this, even the perfectly rational agent with full computational powers qua Turing machine will still arrive at a solution in an adaptive way using trial and error techniques.¹⁰ As with the above cognitive problem of classification, solutions once learnt by trial and error may be retained and passed down in some hereditary process. System-wide adaptive shared operational schema can emerge and these can then be called upon habitually when certain *if-then* conditions are satisfied. Inductive inference or behaviour of this kind which requires no *de novo* solutions is instinctive, atrophied into skilful behaviour within an adaptive shared schema and may in the words of Gode and Sunder (1993) display zero intelligence.¹¹ Examples will be given below of how shared operational schema that aid in self-organisation especially in socio-economic systems, often themselves involve an adaptive and experientially driven process.

1.2. *Perspectives on Self Organisation*

1.2.1. *Self-organisation in systems with endogenous production of novelty*

Following the von Neumann legacy that transformed the foundations of dynamics from energy and motion to information and computation, Wolfram (1984), in what is now called the Wolfram-Chomsky schema, postulates that on varying the computational capabilities of agents, different system-wide dynamics can be generated (see, Wolfram, 1984, Dawid, 1999, Foley, in Albin, 1988, pp. 42–55, Markose, 2001*a*). Finite automata produce Type I dynamics with unique limit points or homogeneity; push down automata produce Type II dynamics with limit cycles; linear bounded automata generate Type III chaotic output trajectories with strange attractors. The significance of this schema is that it postulates that only agents with the full powers of Turing machines capable of simulating other Turing machines, which Wolfram calls *computational universality*, can produce Type IV irregular innovation based structure changing dynamics associated with evolutionary biology and capitalist growth.¹² However, what appears mysterious is why agents with the highest level of computational intelligence are necessary to

¹⁰ In other words, the popular assumption that agents need to have some arbitrary bounds on their computational capacity to justify the adaptive methods of Table 1 Column III, may be a mistaken one.

¹¹ Gigerenzer and Todd (1999) made an extensive study of what they call 'an adaptive toolbox' of fast and frugal heuristics. The claim is that these help to make us 'smart' and *ecologically rational* (p. 13). Interestingly, Gigerenzer and Todd (1999, p. 32) single out the problems highlighted above that have contrarian or oppositional structures. They claim that decision problems with hostile agents and the Arthur minority game structure are ones where agents will resort, respectively, to heterogeneous and unpredictable strategies. However, they give no theoretical or mathematical reason for why these problems pose deep challenges to the standard optimising and computability assumptions made in the mainstream rationality paradigm in economics. Section 1.2.1 followed by Sections 2.2 and 2.3 of the overview to this Feature will give some original insights to this.

¹² It is a matter of controversy as to what conditions produce computational universal agents. While there is agreement that far fewer than the 29-state cellular automata that von Neumann first considered could lead to structures identifiable with a universal computer, there is some doubt whether this is possible in the 2-state cellular automata that Wolfram has considered (Bak, 1996, pp.106–7). The Langton (1990) experiments with a 5 state cellular automata appears to produce Type IV dynamics.

produce innovative outcomes in Type IV dynamics. The legacy of von Neumann towards what Goldberg calls a 'computational theory of actor innovation' (see, footnote 2) makes a veritable break with traditional Darwinian tenet that the source of innovation or mutation in the system is random. Section 3 will indicate that highly commonplace oppositional structures that proliferate in society and immune systems are the clue to understanding what agents, if they were Turing machines, would do.

What is remarkable is that the formal character of equilibria of systems capable of the endogenous production of novelty-based complex dynamics corresponds to the notion of recursively inseparable sets first defined by Post (1944) in the context of algorithmically undecidable decision problems. It is in this context, that Casti (1994, pp. 143–9) makes the connection between complex undecidable dynamics and 'surprises'. Langton (1990, 1992) identifies the analog between the Wolfram-Chomsky complexity classes, the halting problem and the phenomenon of phase transitions and colourfully refers to the phase transition associated with Type IV dynamics as 'life at the edge of chaos'. The recursively inseparable sets are two disjoint sets.¹³ The one on which Turing machines halt is associated with Type I and Type II dynamic systems with limit points and limit cycles, which Langton calls (computable) order. The other set on which Turing Machines can be logically deduced to be incapable of halting represents systems with Type III chaotic dynamics. Both of these sets represent attractors for dynamic systems that cannot produce novelty. The domain for novelty-producing Type IV dynamics lies outside both these sets.

Figure 1 gives the set theoretic representation of the Wolfram-Chomsky schema of the complexity classes for dynamic systems which formally corresponds to Post's set theoretic proof of the Gödel Incompleteness Result (Cutland, 1980; Markose, 2001*b*, 2004).

It is indeed the case that the necessity for self-organisation of a system capable of endogenous novelty production follows from this axiomatic proof on the impossibility of its algorithmic implementation. No finite meta model can ever computably identify the novelty-based change in the structure of this system. As will be seen in Section 4, this has an empirically testable hypothesis well known to economists as the Lucas Critique (Lucas, 1976) on the failure of econometric models to identify structural breaks that follow from strategically induced innovation.

Langton (1990, 1992) makes an important observation that systems capable of endogenous novelty generation experience a critical slowing down at the phase transition between the two other domains of (computable) order and chaos as the system at this juncture is effectively involved in an irreducible process of calculation of an undecidable global ordering problem of Gödel's Diophantine degree of complexity. An important aspect of the non-recursive reordering or self-organisation that is going on when innovations have to be incorporated into an extant

¹³ Technically, these disjoint sets are recursively enumerable but their complements are not. If this were not to be the case, the system will be complete in that no novel encoded objects (not already in these sets) can ever arise.

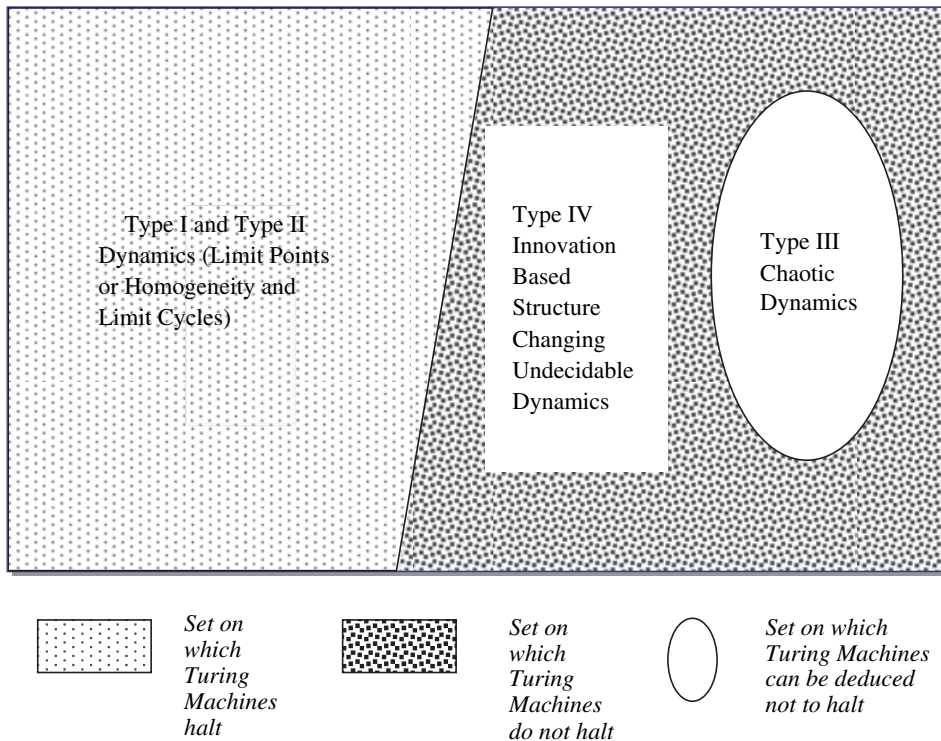


Fig. 1. Set Theoretic Representation of Self-Organisation With Endogenous Novelty

system is that the logical consistency of the internal rules/programs governing the dynamic system must be satisfied at the phase transition.

1.2.2. *On Self-Organised Criticality (SOC) with exogenous innovations: connectivity and networks*

Bak and his coworkers (Bak *et al.*, 1988; Bak and Sneppen, 1993; Bak, 1996) have been very influential in associating some universal macroscopic properties, in particular, the statistical distribution of power laws in systems with large ensembles of micro interacting constituent parts. Power laws have been observed in natural, artificial and social systems ranging from earthquakes and the size of cities to income distribution. The discovery by Pareto (1897) of the power law exponent of 3/2 for wealth distribution,¹⁴ which has stayed remarkably constant over two centuries for capitalist countries, is perhaps the most famous of such laws.

Bak postulates that power laws are the statistical consequence of systems that are capable of self-organisation at critical points. Bak (1996, p. 112) clearly would like to identify his principle of self-organised criticality precisely with systems that have computational universality, viz. those capable of endogenous production of

¹⁴ The probability distribution (or the proportion of individuals in a population with wealth of size w) is given as $P(w) \sim w^{-1-\alpha}$. Here, $\alpha \sim 3/2$ is what is referred to as the power law exponent. See, also Durlauf, in this Feature, for more economic examples.

novelty. In this framework, a sub critical system will converge to homogeneity and a supercritical system will manifest chaotic dynamics with critical behaviour lying between the two as laid out in Figure 1. Despite, the precise mathematical characterisation of Langton's 'life at the edge of chaos' as the critical phase transition associated with Type IV complex dynamics, the latter has unfortunately been the subject of much pastiche, invoked even in the case of systems without endogenous capabilities for generating novelty.

The nature of dynamics associated with self-organisation is qualitatively well known as irregular 'waves' or avalanches' observable from the process of spontaneous reorganisation in capitalist creative destruction and also from records of mass extinctions in evolutionary biology. The Bak hypothesis is that in SOC systems, the number of individuals involved in each wave of birth/death type reorganisation that follows from a 'mutation' in the system will follow a power law. This power law exponent should also be invariant across different scales of population size when the system is critical.

The deficiency, however, with the computer simulation models that Bak and his coworkers have devised, the so called sandpile model and the Bak and Sneppen (1993) model on waves of evolutionary extinctions, is that the systems *per se* do not have any capacity to produce innovation endogenously. In the simple version of the Bak and Sneppen (1993) model with species arranged on a one dimensional lattice, the least fit species (f_{\min}), where fitness is a number between 0 and 1, is exogenously subjected to a random mutation and so are the two adjoining species with their new fitness levels being drawn from a uniform distribution. This fixed dynamic rule when iterated a large number of times produces burst like avalanches. A critical value, f_c in the limit is identified for the level of fitness in the population below which the probability distribution of fitness falls to zero and above it the distribution converges to a fixed value.¹⁵

The Ising model in statistical mechanics adopted and extended by Ligett (1985, 1997) is the fundamental prototype for the dynamics arising from large numbers of local interactions. However, Ligett (1985, 1997) makes no claims for self-organised criticality, Type IV dynamics, or for power laws. Further, for reasons that will be obvious, it will be useful to refer to Ligett criticality (to distinguish it from Bak SOC) as the point at which systems tip in one direction or another. Typically, in the Ligett/Ising framework the diffusion of a binary process is considered, such as infection or no infection in epidemics, to buy or sell in a herding model pertaining to markets, choice between two competing products etc. It can be argued that in the absence of endogenous scope for novelty, there is little to differentiate mathematically between Bak criticality and Ligett criticality except the focus of analysis. Thus, notwithstanding the controversy on whether the agency of change can be assumed to be exogenous, as has been expertly summarised in

¹⁵ For any given level of fitness $0 < f^* < 1$, an avalanche of duration T is defined as follows: for $t < t' < T$, $f_{\min}(t) > f^*$, $f_{\min}(t') < f^*$, $f_{\min}(T+1) > f^*$. That is, the duration of the avalanche is the time T for which the species with the minimum fitness f_{\min} in the population starting from the point at which it exceeds the given level of fitness f^* , falls below it and then recovers again. Bak and Sneppen (1993) find that for f_c the probability density function of the so called avalanches of different durations, has a power law distribution, $P(T) \sim 1/T$.

Jensen (1998, p. 126–7), the two necessary and if not sufficient conditions in such spontaneous reorganisation with critical power law behaviour is interconnectivity in local rules of agents and the application of thresholds. Further, some systems with large numbers of interacting elements may also have network effects which form positive nonlinear feedback loops resulting in increasing returns to scale.

The crucial insight here is on the connectivity of agents in networks and that criticality in systems may also be viewed as those displaying specific classes of patterns of connectivity or networks. In graph theory nodes/vertices can be represented as agents and edges as connective links with a function (fixed or time varying) measuring the intensity or frequency of interaction between (i, j) agents.¹⁶ On the one extreme there are networks where every player can interact with any other (a zero dimensional one) or one where a player can interact with its nearest (k) neighbours. The small world networks (Watts and Strogatz, 1998; Watts, 1999) which are meant to be characteristic of socio-economic networks, is one in which the fixed k -neighbourhood of an individual is modified by breaking a fraction of its k original links. An equal number of new links are created by adding to an individual's neighbourhood a set of randomly selected individuals from the whole system.

As with horses for courses, economic applications of local interacting agent models have taken two types of perspectives. The first perspective favoured by econo-physicists and some ACEs has gone down the Bak route with the focus on self organised criticality and power laws. Here, the Lux and Marchesi (1999) stock market model with local level fixed non-adaptive interactions producing fat tail global price dynamics has been influential. In a recent application of the Bak-Sneppen type framework for the study of technological diffusion, Arenas *et al.* (2002) consider changes driven by cumulative social interactions spreading through social networks with the threshold cost of adoption being dependent on the degree of incompatibility with neighbours. Agents are placed on a one dimensional lattice and an exogenous selection is made of agents to have their technological competence enhanced by a random amount in periods when diffusion waves have died out. Arenas *et al.* (2002) define a parameter k which reflects the relative incompatibility costs of being either ahead of one's neighbours or behind them. Each agent plays a myopic best response as in a coordination game with the important proviso that agent i 's game with $i - 1$ is not independent of what he plays with $i + 1$. They find that below a critical value for k , denoted by k^* , the system shows complete homogeneity in technological competence used and at k^* a critical degree of heterogeneity persists with power laws observed in the size of avalanches. An avalanche is defined by the number of agents who revise their technological level upwards after a random update on the technology of one agent. It must be noted that Arenas *et al.* (2002) do not show self-organisation with k^* being obtained by an endogenous process. They give a series of simulations of the system dynamics with different values of k and report the critical k^* as one for

¹⁶ Mathematically, Green (1994) has found that the many different representations of interconnectivities in systems such as in matrix models (for example, Markov processes), dynamic systems involving differential equations and cellular automata are all isomorphic to directed graphs.

which the size of avalanches follows a power law with an identical exponent irrespective of scale of the population. This is not satisfactory along with the pastiche on Langton's characterisation of Type IV dynamics for their system with no endogenous capabilities for innovation. However, what is important to note is that the critical heterogeneity in adoption strategies that arises from the power law characterisation of self-organising systems is in sharp contrast with the bulk of extant evolutionary game theory models on coordination games. Convergence to homogeneity of strategies in the latter appears to be the norm in contrast to the rich diversity observed in the real world, Kirman (1992).

A very large group of local interaction economic models (Durlauf and Young, 2000) that comes under the rubric of social dynamics for the study of the diffusion of innovation, information or norms is based on the Liggett/Ising framework adapted for economic analysis by Blume (1993). This framework treats local feedback effects as a stochastic process in which the probability that a given person adopts one of two possible actions, say *A* or *B* in a given period of time, is assumed to be an increasing function of the number of his neighbours who have adopted it.¹⁷ There is also an idiosyncratic factor that reflects agents' preference for *A* or *B* irrespective of other agents. Many of these models assume a fixed network structure, though endogenously changing networks have been considered in Mailath *et al.* (1997) and Jackson and Watts (2001). The impact of the spatial interconnectedness or the network structure on the diffusion dynamics has been analysed on a case by case basis by Goyal and Janssen (1996), Chwe (2000) and Morris (2000). The state of the system at each time period is the vector of *A*s and *B*s adopted by each of the agents who are represented by vertices of the graph defining the social network. The potential of a state is the linear combination of the payoffs from the *A*-region, the *B*-region and the idiosyncratic component. Following the Ising model and statistical mechanics, the long-run relative frequency of each state is given by the Gibbs distribution which is concentrated on the states with high potential. Such states have been defined to be stochastically stable by Foster and Young (1990).

As already noted, in contrast to the Bak type criticality literature which studies the critical state of the system when both *A* and *B* coexist, the tipping style Ising models of socio-economic dynamics have focused on asymptotic stochastic stability analysis. The bulk of what passes as complex systems theory currently is the local interaction models of systems that are not capable of producing innovation or ones in which innovations are exogenously introduced. The network dynamics for 2-dimensional Ising model and some of its hierarchical extensions are fully solvable with many of the economic applications here being fully analytical. However, it is

¹⁷ A well-known formulation of this in the stock market context, Sornette (2003), is for an investor, *i*, to do what the majority of his neighbours did in the last period (sign assumes '+' for majority buys and '-' for majority sells) up to an idiosyncratic noise term ε . Thus, $s_i = \text{sign}(K \sum_{j \in N_i} s_{j-1} + \varepsilon)$. The factor *K* is inversely related to market depth or the trading volume and N_i are those investors directly connected to the *i*th investor. The relative strengths of the cohesion/imitative parameter *K* and the variance of idiosyncratic term with a critical value for *K*, defined as K_c , determines whether one way markets form or whether the idiosyncratic noise impedes this. In the critical state there are equal numbers of buys and sells, however, there are large clusters of consensual behaviour with the system being highly susceptible to unstable crash type behaviour.

far from the case that in the models surveyed here critical behaviour is self-organised. It is increasingly being understood that the self-organisation in a system requires endogenising the scope for competitive coevolutionary dynamics referred to as the Red Queen principle which maintains the system at the critical region. This will be addressed in Section 3.

1.2.3. *Self-organisation by evolution of common shared operating schema or institutions*

This section grapples with the idea that on the one hand while cooperative behaviour is a strong force of self-organisation, on the other, as seminally mooted by Axelrod (1984, 1987) cooperative behaviour and such societal norms may be an emergent phenomenon rather than arising from rational calculation. It is well known that the two-person one-shot version of the Prisoners' Dilemma¹⁸ leads to a Nash equilibrium outcome that favours the dominant strategy to defect rather than to cooperate. In what Axelrod calls an iterated Prisoners' Dilemma (IPD), players remember others they previously interact with and also some aspects of the prior outcomes. Axelrod invited a number of game theorists to submit strategies that they believe would win the highest scores when playing one another, a pair at a time, in a round robin tournament. So, Computational Experimental Economics was born. On the face of it, given entrenched priors on what one thinks is rational play, the Bayesian strategy with the other player's moves modelled as a Markov process (one of the strategies that was submitted) could be thought to be the winner. Interestingly, the simplest strategy called Tit-For-Tat submitted by Anatol Rapoport achieved the highest average score. In a series of computer simulations of IPD where agents learn to play robust responses using genetic algorithms in repeated rounds,¹⁹ a whole new notion of robustness of rules in coevolving populations of strategies was mooted by Axelrod. Recently, N person IPD has been generalised to include spatial topology in the interaction of agents, especially in a small-world framework. Abramson and Kuperson (2001) found that when the connective properties were altered from a completely ordered lattice to a random network, for fixed payoffs in the game of N -person IPD, the emergent behaviours that it sustained were non-trivially different.

The classic example of the emergence of a common shared operational scheme that then coordinates large numbers of decentralised trading agents in economic systems is the emergence of money as a medium of exchange. Menger (1892) had conjectured that a medium of exchange could have emerged spontaneously rather than by dictates of the sovereign. Marimon *et al.* (1989) consider a trading regime where there is an absence of conditions that permit barter, viz. the double

¹⁸ Purists on emergent outcomes would argue that cooperative behaviour at a collective level should be shown to arise from agents who have no inherent mechanism for cooperation embodied in them. In other words, what the observer would categorise as cooperative behaviour at a collective level should not be in the local objective/payoff function at the level of the individual. An example of this is given in Hemelrijk (1997) where herd behaviour emerges in aggressive individuals.

¹⁹ Based on evidence in Linster (1992), Binmore (1998) has criticised Axelrod's claim that Tit-For-Tat or strategies 'similar' to it dominated the runs in which cooperation emerged in the IPD version with agents selecting strategies using genetic algorithms. Making the notion of strategies that resemble Tit-For-Tat in machine code more precise, Tsang *et al.* (2003) show that though the main Axelrod premise that the majority of the converged runs shows that cooperation emerges, Tit-For-Tat like strategies only account for 20.1% of all strategies and Grim comes in ahead of all strategies at 24.9%.

coincidence of wants. Further, in addition to goods that agents produce and consume (each type of agent exclusively produces one good that he does not consume and consumes a strict subset of others) there is one good that they neither produce nor consume, say good 0. The experiment uses artificially intelligent agents who meet randomly and make decisions to trade, consume or carry forward goods which are acquired only in the expectation that it may lead to more frequent trades in the goods that the agent consumes. Agents use two Classifier Systems with a scoring/credit accounting system to select rules for trading and another one for consumption decisions. Agents can improve welfare by coordinating on the use of good 0 as a medium of exchange. But, there is no *a priori* way of deducing which trading/consumption rules when mutually adopted will lead to this. In the Marimon *et al.* (1989) model, over time it was discovered that the artificially intelligent agents with minimal decentralised information and aiming only that they increase their utility, *inductively* (Column III, Table 1) found rules that led to good 0 being the common medium of exchange.

One of the important insights from multi-agent adaptive simulations such as by Ackley and Litterman (1992) is the inverse relationship between the need for explicit calculation and learning and the evolutionary development of system-wide well-adapted operational schema or institutions. Once the latter are in place agents need to exercise very little intelligence to get things 'right'. What explicit learning was initially needed when fully adapted becomes atrophied into skilled instinctive behaviour. In the Ackley and Litterman (1992) experiments, agents have neural network brains which are capable of being universal function approximators with the same powers of calculation as Turing Machines. The apparent bounded behaviour in brain activity that is seen *after* the emergence of shared schema should not therefore lead one to conclude that the agents have any *ad hoc* bound to their computational intelligence in the first place.

The recent experimental behavioural studies surrounding the variants of double auctions show that the market in action can produce easy convergence to competitive market clearing prices with very few traders who use a minimum of strategic behaviour or computational powers. In a now celebrated paper, Gode and Sunder (1993) show that continuous double auction markets populated by *zero intelligent* agents are highly efficient in extracting gains from trade and price trajectories converge to competitive equilibrium prices. Zero intelligence corresponds to simple computer programs that generate random bids (or asks) subject to a no loss constraint. The latter means that traders cannot buy above their redemption values or sell below their costs and no attempt is made to maximise profits. This was sufficient to obtain 98% of the gains from trade. Thus, the best price rule of execution in double auction²⁰ is a simple but powerful device to obtain competitive outcomes with the great economies of computation and information that Hayek (1945) emphasised. Sunder (2000) concludes that 'a claim that the predictions of the first fundamental theorem in economics are

²⁰ In a double auction a buyer submits a price at which he is willing to buy (bid) and the seller submits the price at which he will sell (ask). The rule of best price execution states that at any point in time execution of trades occurs at the highest bid in the market and the lowest ask. Those buyers and sellers who cannot meet this are put under competitive pressure.

approachable in classical environments without actual or attempted maximisation by participants might have been met with scepticism until recently. Thanks to a largely serendipitous discovery using computer simulations of markets, we claim that weak forms for individual rationality, far short of maximisation, *when combined with appropriate market institutions*, can be sufficient for the market outcomes to approach the predictions of the first welfare theorem' (italics added) (pp.18–9).

2. The Computation and Identification of Fixed Point Mappings of Market Equilibria

2.1. *The Computational Complexity of Brouwer's Fixed Point and Decentralised Exchange*

Axtell's important contribution to this Feature on the prevalence of decentralised exchange is to show that it overcomes the problem of computational intractability of the Brouwer's fixed point theorem that underpins Walrasian equilibrium models. This is in keeping with the evolutionary solutions discussed in the previous section that enable us to solve problems that may almost surely be beyond methods given in Column I, Table 1. A problem is computationally intractable if it only has exponential time algorithms that have computation times that vary exponentially with the problem size (an integer N), for example, r^N , for some $r > 1$. A polynomial (P) time algorithm is said to be tractable if its computation time varies proportional to the problem size raised to some integer power, d , as in N^d . Exponential functions grow strictly faster than polynomial ones. The significance of P -class problems is that they coincide with those that can be realistically solved by computers and it is held that every practical and efficient algorithm can be rendered as a polynomial time bounded Turing machine.

In general equilibrium theory of markets (Arrow and Hahn, 1971) it was thought that the burden of proof placed by the invisible hand type argument involved the establishment of a *formal possibility* or an *existence result* on an equilibrium in a decentralised economy. Here individuals motivated by self interest and guided by price signals alone are postulated to enjoy consistency of economic plans and Pareto optimal resource allocation. The fixed point theorems of Brouwer and Kakutani that imply generic existence results for equilibrium prices is the mathematical tool that gives credence to the Walrasian model of an exchange economy.

Mathematical problems where a solution is known to exist as in the case of the existence results of Kakutani and Brouwer are called *function problems* (FP). This is in contrast to the structure of *decision problems* where a yes or no answer is sought to whether the problem has a solution. Typically, for function problems more elaborate output than 'yes' or 'no' is required. Axtell, in this Feature, uses recent results on the complexity class of Brouwer type function problems for fixed points (Papadimitriou, 1994) which show them to be related to that of the parity argument via the Sperner's Lemma.²¹ The parity argument arises from a graph

²¹ Sperner's lemma states that *any admissible colouring of any triangulation of the unit triangle has a 3-coloured triangle*, (viz. an odd number of them by the parity argument). While a directed graph to the 3-coloured triangle (within the larger triangle) exists, the constructive problem is that there is as yet no known polynomial time algorithm for this.

theoretic lemma that any finite graph has an even number of odd-degree nodes. Sperner's Lemma guarantees the existence of a directed graph to the 3-coloured triangle identical to the larger one, but to date no known polynomial algorithm exists for this. The analogue to Brouwer's fixed point theorem follows from the generalisation of Sperner's Lemma when applied to a N -dimensional simplex guarantees the existence of a panchromatic simplex. The Brouwer's fixed point function $f(x^*) = x^*$ can be proved to exist at the limit of the centre, x^* , of panchromatic simplices obtained by finer and finer triangulations, (Papadimitriou, 1994). Once again, due to the above analogue to Sperner's Lemma, there are no known polynomial algorithms for the Brouwer and Kakutani fixed points specifically in the general case of nonlinear utility functions for the Walrasian equilibrium model. Axtell concludes that the exponential (non-polynomial) complexity of the Walrasian fixed point algorithms as opposed to polynomial computations involving N , the number of commodities in the economy, may rule them out as a plausible computational device for equilibrium price formation in real world markets.

Axtell conjectures that polynomial complexity that is a prerequisite of real world equilibrium price formation mechanisms may crucially arise from institutional arrangements that promote informational and computational decentralisation. The aggregation of information into the net demand functions of the Walrasian model has long been known to be excessive. The much quoted observation regarding this being that of Hayek (1945, italics added) 'We cannot expect that this problem will be solved by first communicating all this information to a central board which, after integrating all knowledge, issues its orders *The problem is to show how a solution is produced by interaction of people each of whom has partial knowledge*'. (p. 524)

However, it is not till very recently that it has been recognised that the invisible hand type arguments also involves a burden of proof along the lines of what Rust (1998) has called *computational decentralisation* that militates against the highly aggregative calculation involved in the fixed point algorithm of the Walrasian model.

Axtell, in this Feature, suggests that instead of imputing the locus of equilibrium price calculations to reside in an entity such as the Walrasian auctioneer, it is more appropriate to consider the decentralised market as a whole as a collective computing device. Axtell then proceeds to show that parallel distributed agent-based models of k -lateral exchange can restore the all important polynomial complexity in calculation of equilibrium prices and allocations.

2.2. Adaptive/Inductive Learning of Rational Expectations Equilibria

A major part of the coordination processes in markets rely on the identification and calculation of fixed point mappings of global signals such as equilibrium prices. By forming rational expectations of fixed points of market equilibrium prices economic agents are meant to coordinate their activity and iron out inconsistent expectations. Spear (1989) was the first to show that the problem of identifying a set of fixed points for market equilibrium price functions is

algorithmically unsolvable.²² The absence of a unique decision procedure or a deductive means by which to select appropriate forecast models/functions has been intuitively justified as arising from the self-referential character of the problem by Arthur *et al.* (1997): ‘the expectational models investors choose affect the price sequence, so that ... their ... very choices of models affect their data and so their choices of model’. Arthur *et al.* (1997) refer to the above self-referential structure of the problem as causing it to be *ill defined*. Likewise, in the El Farol game, Arthur (1994) gives the classic prototype of a problem with a contrarian structure for which there is no deductive means of arriving at a solution. The methods of Table 1 Column II can show precisely how these problems are well understood to be algorithmically unsolvable. Mathematically to solve such problems, methods of inductive inference including that of heterogeneous adaptive agents of Column III, Table 1 are required.

Arifovic (1994) pioneered the first adaptive social learning model of a homogeneous rational expectations equilibrium using a single population genetic algorithm. However, in the El Farol game of Arthur (1994) where punters at a popular bar like to be there on days when it is not crowded, the very notion of homogeneous rational expectations is inappropriate to the problem. The El Farol game presents a commonplace preference to be in the minority or a situation when pay offs are greatest whilst being contrarian. An example of this is to be a buyer (seller) when the majority is selling (buying). It is intuitively clear that if all agents share the same forecasting model, then actions taken by agents which are consistent with the forecast will defeat their objective to be in the minority. The fixed point of such a game involving a contrarian structure represents a non-computable fixed point, see Markose (2001*b*, 2002, Section 2). In the absence of computable fixed points, rational agents even with the same information must agree to disagree. Indeed, in the large literature that has since grown on the Minority game (Challet and Zhang, 1998) the self-organised equilibrium that arises in the absence of computable homogeneous rational expectations is sustained by a critical degree of heterogeneity in strategies. This self-organised equilibrium attains greater Pareto improving welfare gains than the Nash equilibrium outcome in which agents randomise.²³

In the Arthur *et al.* (1997) agent-based stock market model all the standard neoclassical features with preferences, constraints and even optimal portfolio choice rules appear. The agents are however heterogeneous in their forecast models for asset returns and have to use adaptive GAs/GPs of Table 1, Column III as deductive means do not exist to solve such strongly self-referential mappings.

²² In the computability framework, rational expectations involves an executable meta forecast function whose code represents a fixed point of the market clearing price function. This has been precisely stated in Markose (2002, Section 2).

²³ Denoting all N traders as buyers, N_b , or sellers, N_s , the ‘social’ gains from trade are optimised when after some large number of trading periods, T , the following variance function $\sigma^2 = (1/T) \sum_{t=1}^T (N_{bt} - N/2)^2$ is at a minimum. What is interesting is that the mixed strategy Nash equilibrium where agents randomly buy or sell with probability half is not in fact Pareto optimal or the emergent outcome. There is a generic degree of heterogeneity of strategies for any N for which the σ^2 function is minimised as the agents inductively solve the problem. In the latter, the minima for σ^2 via self-organisation are below what is obtained by the Nash equilibrium random mixed strategies.

The objectives of the agent-based stock market experiment were to study the conditions under which homogeneous rational expectations of stock returns follow and whether the stylised facts of fat tails and volatility clustering can be reproduced. As the market environment is maintained relatively stationary with a low intensity of search by GAs or GPs for better trading strategies, the unique homogeneous rational expectations result followed in Arthur *et al.* (1997) accompanied by a cessation of trade as in the no trade theorem. In contrast, when the rate of GA exploration for 'better' predictors was sped up, the stock market prices began to show some of the historically observed properties of volatility clustering and crashes. However, as will be pointed out in the next section, in so far as the rate of performance enhancement, the timing and extent of retraining of GA/GP agents were exogenously altered by the experimenter rather than endogenously determined in terms of the Red Queen dynamic, it is difficult to talk about (Chen and Yeh, 2001) the self-organised or emergent nature of market efficiency.

3. Evolution of Complexity and the Red Queen Principle

3.1. *The More Things Change, the More They Stay the Same*

The Red Queen principle has been espoused by both evolutionary biologists (van Valen, 1973) and physicists (Anderson, 1977) to characterise a relentless arms race type dynamic of change among competing elements at a micro level with remarkable macro level stability of the system along some dimensions. Conveying these two aspects of the system dynamic, Anderson in his Nobel Prize lecture (1977) said 'it is remarkable that in almost all cases interactions play a vital role, yet many results are not changed too seriously by them. ... We have gone to extraordinary lengths to make our magnetic moments ... or electrons... *to stay in one place* (italics added) ... This situation was foreshadowed by an eminent nineteenth century mathematician named Dodgson,' (p. 397). The latter, writing as Lewis Carroll, has the following passage in his book *Alice Through the Looking Glass* which has lent itself to the so called Red Queen principle: "Well in *our* country," said Alice, still panting a little, "you'd generally get to somewhere else if you ran very fast for a long time as we've been doing." "A slow kind of country!" said the Red Queen. "Now here, you see, it takes all the running you can do, to be in the same place."

In Physics, the Red Queen principle asserts that a continuous application of offsetting or contravening force at a micro level can maintain global stability. In evolutionary biology, the rivalrous coevolution of species is seen to be a spur to the evolution of complexity itself, manifesting, however, no net gains in relative fitness. Here, we will be guided by Robson (2002) who is among the first to have discussed the Red Queen principle and its relevance for economics. Robson represents an economist who has taken the time to study the evidence for competitive co-evolution in biology to see what implications it may have for the growth of complexity in socio-economic institutions and market environments. He is of the view that the Red Queen type competition among species drives the most remarkable of macro evolutionary phenomena, viz. the emergence of innovative forms and more complex organisms. Analogous to this in the economic domain is the Schumpeterian

model of capitalist growth which is characterised by monopolistic firms which develop innovative products involving new technology as barriers to entry for competing firms. The static world of Welfare Theorems of perfect competition with fixed preferences and technology – which many will accept as being relevant for local optima – cannot adequately deal with macro evolutionary trends that account for movements to global optima and the production of new and more complex forms. Such dynamic competition, which Robson thinks is bound to be more important than the precepts of static efficiency, still remains outside the ambit of much of standard economic theory. Indeed, in the light of vast evidence from evolutionary systems of the growth of complexity, Robson, in his masterful survey in this Feature, concludes that it is striking that much of extant game theory overlooks the Red Queen principle and appears to argue that there is no such arms race in complexity. There is not only little evidence that evolutionary systems select *against* complexity, as we will see, there are compelling reasons why an arms race in complexity is inevitable among intelligent agents.

In the Red Queen effects associated with the notion of co-evolving populations of species, the furious pace of ‘running’ relates to each species attempting to enhance its fitness relative to others. In the competition that governs the fight for scarce resources or in cases of direct confrontation with zero sum payoffs such as in parasite-host or predator-prey situations what matters is relative rather than absolute performance capabilities of the individuals. Certain attributes of individuals have to be enhanced relative to the same in others to maintain *status quo*, let alone to get ahead of the game. As with one, others are like wise involved in performance enhancement: triggering off an arms race. Since only hypothesis and conjectures but not direct tests of the Red Queen principle can be applied to evolutionary biology, Artificial Life simulations have become the means to understand the system dynamics and the distinctive features of competitive co-evolution. The classic work of Ray’s (1992) Tierra and Sim’s (1994) creatures are based on the principle of competing coevolving species. In Hillis (1992) parasites were deliberately introduced and it was noted that competition among coevolving species could potentially prevent stagnation at local optima. Robson, in this Feature, gives an analytical example of how this might be the case. The evolution of intelligence itself is hypothesised to arise as an arms race giving rise to Machiavellian behaviour in social interactions *within* the same species, Robson (2002).

Robson, in this Feature, also gives a detailed account of how heterosexual sex which is instrumental in giving ample variation to genetic material of the species can be interpreted as a Red Queen type response to evade parasitic behaviour of micro organisms that can themselves mutate and resist antigens produced by host species.

3.2. *Some Economic Examples of Red Queen Effects*

What significance might the Red Queen principle have for economics?

The Red Queen effect can manifest in the arms race of product innovation among firms in capitalist systems, simply for them to maintain *status quo* in market shares. In recent unpublished extensive work on a cross section of industries by

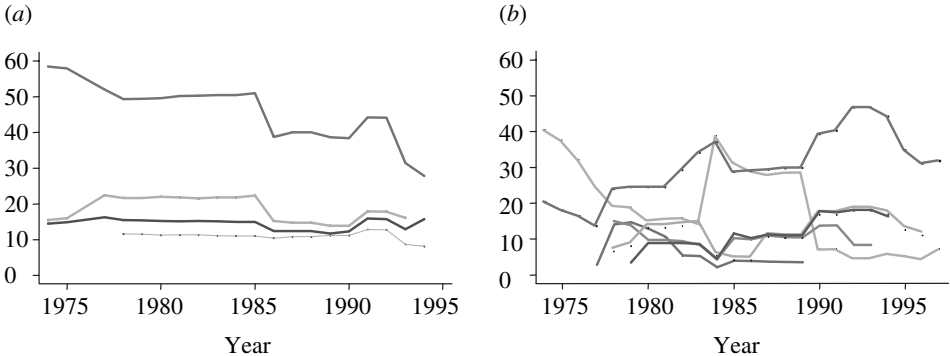


Fig. 2. Market Shares: (a) Margarine; (b) Cash Registers

Sutton, he finds strong evidence for this form of non-price competition. In the case of a number of industries or product categories where the leading firms were locked in a prolonged arms race of product innovation, market shares were remarkably stable, viz. rival firms' efforts offset each other. Innovating firm's products are either imitated by the rival firms or countered by their own innovations. However, there are also examples of rivalrous product innovation where the brand leaders are overtaken and market shares are volatile. Failure to respond quickly enough to the competition is held up as the reason for the erosion of market share. Figure 2(a) gives evidence from the margarine industry where although scores of new varieties of margarine differing in texture, flavour and hardness were introduced by the firms, their market shares remain more or less unchanged.²⁴ In the case of cash registers, Figure 2(b), where arguably the scope for technological change and product innovation is vastly greater than for margarine, the market shares were highly volatile. For our purposes here either outcome is in keeping with the Red Queen principle in that players cannot maintain *status quo* along a critical dimension unless they can keep up with their peers.²⁵ The interesting question is what global properties a system would exhibit if agents in it were relentlessly trying to 'keep up' Red Queen style.

In financial markets, enormous investment in innovative investment strategies is made to keep up and surpass the average performance of the market as a whole. In Markose *et al.* (2005) the so called Red Queen constraint is endogenously imposed on the forecast and trading performance of a multi population GP agent stock market environment. The premise here is that when all speculative investors are constrained to 'beat' the market, there comes a point of market equilibrium when

²⁴ I am grateful to John Sutton for letting me use his findings and graphs here.

²⁵ Recently, there have been some references to the Red Queen effect in the Economics literature; see Baumol (2004); Khalil (1997); Barnett and Sorenson (2002). We are grateful to one of the anonymous referees for bringing this to our attention. Of these, the work of Baumol (2004) and Khalil (1997) come closest to the way in which the Red Queen effects have been characterised in this Feature, viz. as a dynamic behind innovation and the growth of complex variegation in firm development in capitalist systems. Barnett and Sorenson (2002) make an empirical study of retail banks in Illinois to see if banks involved in a more competitive environment than those that are not, are likely to survive longer.

it is impossible for anyone to do so. Market efficiency is an emergent or unintended consequence with the power law properties of investment income distribution and stock prices that satisfy the stylised facts of fat tails in asset returns.

Red Queen effects also appear in the arena of regulatory arbitrage resulting in arms race in regulatory tightening and creative evasion by regulatees. A plethora of recent institutions (Miller, 1986) have arisen when regulatees who find it profitable to break regulatory structures do so by exiting and innovating new institutions that place them beyond the long arm of the law. The notion that surprises or innovations should be used strategically as best response because anticipated policy may be contravened by innovative regulatees has a long tradition in macro economics literature, Lucas (1972). The next Section will briefly discuss the significance of Lucas's seminal insights on why the structure of opposition such as between regulators and regulatees with agents capable of anticipating the outcomes of others' strategies generates innovative structure changing Type IV dynamics.

3.3. *The Ubiquitous Structure of Opposition and Strategic Innovation*

Innovation is clearly war by other means in market systems, evolutionary biology and immunology which involve oppositional zero sum parasite-host type rivalrous relationships. Ho and Saunders (1986) and more recently Kauffman (1993) have criticised the inadequacy of the Darwinian view that new forms are but a conglomeration of random variations accumulated by natural selection and preserved by heredity. In extant game theory whether deductive or evolutionary there is no notion of innovation being a Nash equilibrium strategy let alone one that is necessitated as a best response by a structure of opposition. As in traditional Darwinian evolution, in economic models innovation is either introduced at a random rate or as an *ad hoc* addition in the form of trend growth.

The missing piece has been the dynamic behind the emergence of variety in new forms. The Red Queen type rivalrous co-evolution accounts for a contextual and strategic explanation for species being finely honed to the characteristics of other rival species. Recent empirical research on RNA viruses which are considered to be the most important group of intracellular parasites has led researchers to hypothesise the notions of hyper-mutability and that of a 'mutator phenotype'. The latter is a behaviour whereby high rates of mutation are adopted with the RNA virus maintaining a highly heterogeneous population called a *quasi species* which generate new escape mutants that can avoid detection from the immune system. In other words, mutation rates are strategically induced rather than arising randomly and this endogenously determined dynamic sustains self-organisation in the critical region.²⁶

The Red Queen principle on having to run faster and faster to survive in the face of rivals, therefore, offers two outward manifestations of an arms race:

²⁶ The RNA virus is able to maintain a critical level of heterogeneity in the *quasi species* with their mutation rates being possible up to the point at which genetic coherence and heredity break down, Solé *et al.* (1997). At the latter point the immune system wins and a heterogeneous environment for the RNA ceases to exist.

- (1) In performance capabilities which can come in the form of enhancement of some *existing* faculty in the species such as intelligence, sensory faculties, speed of locomotion etc.
and/or
- (2) In the use of novelty or surprises viz. the explicit production of new forms.

The ubiquitous structure of opposition envisaged in the Red Queen alone cannot suffice for the emergence of novelty or new features. As noted previously, it has long been conjectured by Wolfram (1984) that it takes agents with the highest powers of computational intelligence that resides in Turing machines and enables them to make simulations of computations made by other agents to produce innovation based structure changing dynamics. Wolfram has conducted extensive computer simulations to test his major hypothesis that computational universality is more generic than hitherto assumed and can arise in simple cellular automata (see, footnote 12). Bio-chemical or some other means by which agents can make self-referential calculations of hostile behaviour of other agents is the trigger for strategic innovation envisaged in these contexts are a far cry from random mutations that have so far been held to be the only means for the emergence of new forms or 'objects'. As yet, it is not fully understood how the immune system bio-chemically detects hostile agents (Holland, 1992). Markose (2002, 2004, Section 3) uses the epochal Gödel (1931) incompleteness result to show that the conditions of opposition between two Turing machines *and* for these machines to mutually recognise this structure are *logically* necessary for innovative outcomes that have an encoding that is beyond algorithmic enumeration. Gödel (1931) had seminally used the notion of the Liar, the agent who falsifies or controverts, to embody the pure logic of opposition. However, the Liar can falsify with certainty only from a computable fixed point. This is intuitively well understood in the Lucasian thesis on policy ineffectiveness that regulatees can contravene policy only if the policy outcomes can be rationally expected. When there is mutual recognition of the players of the structure of opposition, the so called fixed point with the Liar can be fully deduced to be a non-computable fixed point. Any total computable function from this non-computable fixed point, referred to as the *productive function* in Post's set theoretic proof of Gödel incompleteness and as shown in Markose (2004) to represent the best response function, can only map into a set that cannot be algorithmically enumerated with the *productivity* of the function implying an unending succession of such innovations. This coincides with the notion of the strategic use of surprise as intuitively proposed by Lucas (1972) along with the mathematical formulation of an arms race. The corresponding equilibrium Type IV dynamics converges to the non-computable domain between recursively inseparable sets is encapsulated in the famous thesis of Langton (1992) on 'life at the edge of chaos' and has been set out in Figure 1. The undecidable dynamics implies the logical impossibility of meta (econometric) models to identify the innovation induced structure change recursively – a matter once again that has been raised in the Lucas Critique (1976).

It was not till the seminal work of Binmore (1987) that game theory even considered the major implications for dynamic systems and the fundamental

manifestation of indeterminism, as yet not fully understood by economists, of incorporating agents with computational universality and the Gödelian logic involving the Liar or the rule breaker.²⁷ A Nash equilibrium in which agents innovate as a best response to evade hostile objectives of other agents and produce novel objects not previously in action sets is currently outside the ambit of traditional game theory. The latter, without the scope of the mathematics of incompleteness, can only consider randomisation and not innovation in zero sum and oppositional situations. Markose (2001*b*, 2004) shows that it is indeed the Gödelian logic, the capacity for self-referential calculations by an agent of hostile behaviour of other agents, with the ubiquitous structure of opposition formalised in the Liar, that constitute the necessary conditions to trigger innovative behaviour with secrecy and subterfuge as being part and parcel of the rational Nash equilibrium strategy at this juncture. In Ray's (1992) classic artificial life simulation called *Tierra*, in a Red Queen like arms race, when some agents perceive that others are parasitic on them, they start hiding their whereabouts and also mutate to evade the parasite.

4. Empirical Economics, Power Laws and Other 'Stylised' CAS Facts

4.1. *The Identification Problem*

As rightly pointed out by Durlauf in this Feature, many of the precursors in the economics literature on CAS, both in the area of economic history and social interaction models have eschewed any explicit connection to complex systems theory either in terms of von Neumann's grand computational foundations or some subset of it as espoused by econophysicists. Nevertheless, we are particularly indebted to eminent economic historians, institutional and evolutionary economists from Hayek, North to Nelson and Winter to wean us off a mathematical economics that is a far cry from the real world complexity.

Despite claims of universality of certain statistical properties of CAS outcomes, in so far as self-organisation formally requires non-computability of specific end-states, patterns and equilibria, the significance of the system running its course or path dependent outcomes comes to the forefront. Economic historians and evolutionary economists such as Nelson and Winter (2002), which gives a recent overview, have often used their historical case study approach to underscore the point that the problem of path dependence makes the study of processes involved of interest in contrast to end state asymptotic analysis and static efficiency results of neoclassical analysis. Barring Type I and Type II dynamics, the lack of independence from initial conditions and the relevance of the specific time path of processes have to be presumed.

Durlauf bravely tackles the historical accounts on the QWERTY keyboard to address the conceptual issues surrounding path dependence, such as whether

²⁷ It is my view that it is such irrevocably non-computable structures in strategic situations that Simon (1998) calls the 'out guessing problem' which makes the 'whole concept of rationality ill-defined ... and ... [we] must adopt some different framework (outside extant game theory) to explain behaviour under these conditions' (p. 188).

technological lock-ins are inefficient equilibria or are in fact efficient ones given network effects, fixed costs and so on. It has been argued by Axtell *et al.* (2002) on the emergence of equity norms and other path dependent outcomes, that agent models are needed to show in an experimental set up how depending on initial conditions, social systems can manifest meta stable states of inefficient and inequitable social norms that are observed in many societies. A meta stable state is one in which a system spends a very long period of time in terms of intermediate time scales and typically such behaviour of the system cannot be analytically determined, even if as in Axtell *et al.* (2002) the asymptotic stochastically stable outcome can be determined *a priori*.

The bulk of Durlauf's thought-provoking contribution on the extant empirical work on complexity in economics surrounds what he calls the identification problem. Inefficient equilibria as he notes can arise from coordination failure as much as from maladaptive lock ins. Further, there appears to be no consensus as yet as to what causes power law distributions which have been held up as a manifestation of self-organised critical behaviour.

4.2. *Endogenous Economic Explanations of Power Law*

In many econo physics models, power laws as a stylised CAS fact arising from self-organised criticality, as we saw in previous Sections, fall far short of a demonstration of self-organisation. This is because the agency for innovation in the system is external to it. In this Feature, Durlauf also raises the problem of the absence of endogenous economically sensible explanations in some econo physics models. However, as already noted, even agent-based stock market models that have what many would consider to be all the sensible features of neoclassical models, require external tuning to bring about asset price dynamics that correspond to observed stylised facts to do with fat tails and volatility clustering. The latter are anomalous with the standard Gaussian assumptions²⁸ on stock returns and neoclassical models on homogenous rational expectations and no trade theorems.

Lux (2000, 2002) has expended much effort to ascertain the tail index of asset returns in a number of markets. In these studies, Lux finds that values for the tail index for asset returns lies between 2.5 and 4 guaranteeing at least finite first and second moments for the processes. As the latter rule out Levy Stable distributions for asset returns first conjectured by Mandelbrot (1971), given the statistical rejection of null hypothesis of Gaussian/normality for stock returns, it is still an open question as to what class of statistical processes asset returns belong. Further, there is a large body of alleged evidence for persistence or long memory observable as the hyperbolic decay of the auto correlation function of volatility of asset returns (the absolute or the square of asset returns) (Ding *et al.*, 1993; Breidt *et al.*, 1998). Alfarano, Lux and Wagner (2003) put agent stock market models to good use by finding, after painstaking investigation, that the popular herding framework

²⁸ Stock market crashes and other not so extreme events seen in asset markets are anomalous to the Gaussian model of asset returns. In the latter, the 1987 stock market crash with a 22.6% fall in stock returns in one day is an event that is so improbable that it cannot happen even in the lifetime of the universe, see Sornette (2003, p. 50).

of Kirman and Teyssi re (2002), which has previously been upheld as an ‘explanation’ for long memory in asset volatility, can only be a spurious result. This is because analytically, the structure of the herding model with two groups of agents, fundamentalists and chartists, follows a bounded Markovian process with a bimodal limiting distribution and ‘any memory in the system is wiped out by stochastic fluctuations between the two modes of the distribution’. Nevertheless, standard statistical tests for long memory, when applied to simulated data from this class of agent models, erroneously support the hypothesis for power law decay in the autocorrelation function of the volatility of asset returns. Likewise, Durlauf, in this Feature, lists a number of shortcomings in the statistical tests of power and scaling laws so far conducted in the literature on markets as CAS.

A number of econo physicists such as Sornette (1998), Lux and Sornette (2003), Levy and Solomon (1996), Solomon (1998), based on earlier work by Keston (1973) and Goldie (1991), have identified stochastic multiplicative processes²⁹ with non-linear lower bound constraints as being contributory to power law distributions in certain economic models. In the Levy and Solomon (1996) artificial stock market model, this lower bound constraint appears for each agent as a condition that his wealth does not fall below the average investor wealth. Milakovic (2001), in a survey on endogenous explanations for the power law exponent close to $3/2$ in wealth distribution, claims that the condition for power law distribution in the Levy and Solomon (1996) model ‘depends on an arbitrarily imposed lower bound’ constraint on the individual agent’s wealth, defined as a function of the mean wealth of the population. Following recent developments on the Red Queen principle reported in Section 3, Markose *et al.* (2005) have argued that this is far from an arbitrary constraint. It can in fact be viewed as the so called Red Queen constraint. The satisfaction of this lower bound constraint is modelled in Markose *et al.* (2005) as a behavioural one that arises endogenously from coevolving stock market agents. It then dictates performance enhancement in adaptive learning of investment strategies that will enable agents to ‘keep up’ with the market Red Queen style. It has been analytically shown by Solomon (1998) that if no attempt is made by those who fail to keep up, then the crucial power law exponent α on wealth distribution (see footnote 12) can fall below 1, signalling a highly non-egalitarian wealth distribution. Conversely, if ‘too much’ effort is put into keeping up, then the α power law exponent will exceed 2, which manifests a more egalitarian wealth distribution than can be warranted by the historically observed $\alpha \sim 3/2$. We have seen that in artificial stock market models in which agents have to learn and adapt individually in a multi population GP environment, retraining of GPs is mostly done in an *ad hoc* way. When the Red Queen constraint was explicitly set up in a multi-population form of constraint enhanced GPs, who are the agent/investors in an artificial stock market, Markose *et al.* (2005) find some evidence on how the crucial $\alpha \sim 3/2$ can emerge in investor wealth distribution. When the Red Queen constraints are not imposed in the artificial stock market, the power law exponent on investor wealth diverges markedly from the required $\alpha \sim 3/2$.

²⁹ A stochastic multiplicative process is one where $x(t + 1) = \lambda x(t)$, with λ random, even time varying with finite support.

However, these results are preliminary and it is far from conclusive whether what one might consider to be CAS type agent based features can endogenously in a behavioural sense explain the regularities in the global data.

4.3. *Econometric Models of Surprises in Policy Games and the Lucas Critique*

One of the major themes of the computational legacy of CAS as discussed in Section 3.3, is that the two Lucas (1972) postulates on policy ineffectiveness, the strategic use of surprises and the so called Lucas Critique, Lucas (1976), are stylised CAS facts of fundamental significance.

In the extant policy design literature, the nexus between predictable policy leading to policy/institutional failure and the role of surprise strategies is confined to highly stylised macro economic models on monetary neutrality and surprise inflation.³⁰ Remarkably, though macro-economists in the last three decades made the notion of 'surprise' strategies common currency, at no time was it regarded necessary to establish a generic class of (policy) games where 'surprise' strategies hold in a Nash equilibrium in accordance with the first Lucasian postulate on why policy is rendered ineffective. Hence, the generic necessity of the strategic use of indeterminacy due to the presence of hostile agents has not been recognised in the macro policy literature.³¹ Further, the indeterminacy of optimal behavioural structures raised in the problem of lack of structural invariance in the Lucas Critique is regarded to be unrelated to the first two Lucasian postulates.

Some of the econometric issues relating to the Lucas Critique were recently reviewed in Hoover (1995). Indeed, many (Ericsson *et al.*, 1995, 1996) hold that the applicability of the Lucas Critique on the lack of structural invariance is an empirical question, refutable by the presence of invariance in the system.³² Thus, it is not far off the mark to say that it has not been recognised that theory, in particular, CAS theory can throw light on this endogenously arising innovative behaviour of agents as best response strategies in rivalrous oppositional structures which then lead to Type IV undecidable dynamics of Figure 1.

With no theoretical guidance as to which class of games warrant elimination of extant policy rules/institutions and the adoption of innovative or surprise strategies in the Nash equilibrium signalling structural breaks in the system and therefore to predictive failure with meta forecast models, Ericsson and Irons

³⁰ The notion of a surprise strategy in the economics literature appears in the so called Lucas surprise supply function often defined as follows: $y = y^* + b(\pi - \pi^e) + \varepsilon$. This says that output, y , will not increase beyond the natural rate, y^* , unless there is 'surprise' inflation, $(\pi - \pi^e)$ which is the white noise prediction error from expected inflation, π^e . The idea here is that the private sector contravenes the effects of anticipated inflation, viz. the neutrality result. Hence, it is intuitively asserted that authorities who seek to expand output beyond the natural rate need to use surprise inflation.

³¹ The attempts by macro policy theorists to eliminate 'surprise' inflation and indeterminism by using a transparent precommitment strategy in the form of the currency peg led to some of the worst policy disasters with the serial collapse of pegged systems in the last decade, Eichengreen (1999). Soros (1995) allegedly claimed that he used the Liar strategy to profit from the collapse of the British currency peg in the summer of 1992. None in the macro economics fraternity saw the relevance of Binmore (1987) who raised the spectre of Gödel and hence the significance of the allusion to the Liar/hostile agent who will bring about the demise of institutions that are based on strategies that have predictable outcomes.

³² I am grateful to L.Ellis from the Reserve Bank of Australia for bringing this to my attention.

(1995) found ‘exactly zero papers (that yield) empirical support for the Lucas Critique’ after an extensive survey of the empirical literature, Leeper (1995, p. 314). Finally, it must be noted, adaptive agent models of Column III, Table 1 are methodologically better suited to studying unintended consequences of changes in policy rules as they are designed to manifest co-evolution and do not run into the Lucas Critique of econometric models.

5. Concluding Remarks

Despite the deep mathematical foundations of CAS, remarkably, systems capable of adaptive novelty are commonplace and only involve the intuitively familiar notion on the need to evade hostile agents. Axelrod (1987) in his classic study on cooperative and non-cooperative behaviour in governing design principles behind evolution raised the crucial question on the necessity of hostile agents: ‘we can begin asking about whether parasites are inherent to all complex systems, or merely the outcome of the way biological systems have happened to evolve’ (p. 41). Based on the Gödel (1931) incompleteness result, in addition to the formal structure of opposition as being a logically necessary condition for evasive and innovative behaviour, Wolfram (1984) had conjectured that the highest level of computational intelligence, the capacity for self-referential calculation of hostile behaviour, was also necessary. The role of economists in the multi-disciplinary edifice of CAS, especially their contribution to the crucial insights and ingredients of CAS theory on the strategic use of surprises and indeterminacy as part of successful evolutionary design principles, is not insubstantial.

It is increasingly being understood, as explained in Sections 1.2.1 and 3.3, that the seminal Bak notion of self-organised critical systems with large numbers of interacting constituent parts that sustain a critical degree of variety or heterogeneity, requires an endogenous dynamic currently being referred to as the Red Queen principle. The latter can be viewed as governing the rate of performance enhancement and/or the rate of innovative activity among co-evolving competitive agents. There is however, no consensus as to whether it is the Red Queen principle and/or some other endogenous causal factor, such as the diffusion of information in networked systems, that gives rise to the power laws associated with SOC systems. Indeed, as we have noted, many socio-dynamic models have eschewed any CAS lineage as we have yet to understand fully the notion of self-organisation with a critical degree of heterogeneity to supplant homogeneity and also the universality principles on the network connectivity in agents to maintain criticality. Durlauf, in this Feature, has given further caveats on the problems of testing stylised CAS facts.

The new computational CAS paradigm clearly surpasses the traditional evolutionary Darwinian framework of random mutation, heredity and natural selection. The principles of self-organisation outlined in this overview go well beyond the Darwinian ideas of natural selection currently operationalised in replicator dynamics. Crucially, the formal results that agents with the highest level of computational intelligence are needed to produce adaptive novelty with empirical evidence of the Red Queen dynamic in immunology, in product innovation and regulatory arbitrage in the capitalist system, has cast doubt on random mutation as

being the only source of variety. The equilibrium with self-organised complexity with adaptive novelty of Figure 1 has the highest degree of algorithmic unsolvability of the Hilbert Tenth problem. This model of indeterminism is a far cry from extant models that appear to assume adaptive innovation or strategic 'surprise' (see, footnote 27) as white noise which in the framework of entropy represents perfect disorder, the antithesis of self-organised complexity. It can be conjectured that a lack of progress in our understanding of market incompleteness and arbitrage free institutions is related to these issues on indeterminism.

This Feature on markets as CAS has been critical of some multi-disciplinary perspectives of complex systems theory to economics. It is also original in highlighting the computational foundations of CAS theory and the significance of the Red Queen dynamic in the shaping of macro evolutionary trends and in the process of self-organisation that governs the critical degree of variety or heterogeneity in systems. We argued that mathematically, problems involving strongly self-referential global/system-wide mappings, like an endogenously determined price or reward system, or those that involve coevolutionary contrarian or hostile agents make these impossible to solve by deductive means of Column I, Table 1. Adaptive/evolutionary methods of Column III, Table 1 have to be used and models beyond optimisation and the rational economic man of neoclassical economics are needed. Methodologically, the view of CAS as a novelty producing system with attendant problems of incompleteness spells a paradigm shift for many mainstream economists who have operated under the assumption of closure and completeness. The constructive dimension of agent based models has opened up a new branch of 'computational mechanism design' which utilises the computer environment for 'wind tunnel tests'. More ambitiously, the principles of self-organisation or artificially evolved solutions have begun to feature in the design of markets and also of policy. The pragmatic aspects of these developments will accelerate the assimilation of this new field into economics.

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