A Theory of Money Creation by Banks and Central Banking in a Two-Date Economy

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Abstract

This paper considers money creation by banks and central banking in a model where a means of payment is issued by both the central bank and banks, and the private issuance is endogenous in competitive equilibrium. The economy lasts for two dates, but the central bank gets its purely nominal issues circulated. In circumstances where banks issue too little money, quantitative easing policy improves efficiency by substituting inflation for bank default as the way to make the real value of money contingent on a productivity shock. This paper also considers interest-rate policy and leverage regulation. The importance of competition between banks is underlined.

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The community cannot get rid of its currency supply... The "hot potato" analogy truly applies. For bank-created money, however, there is an economic mechanism of extinction as well as creation, contraction as well as expansion. – *James Tobin (1963)*

1 Introduction

This paper presents a new approach to consider central banking in relation to the circulation of fiat money. It has three features. First, it builds on an competitive-equilibrium analysis of money creation by private-sector banks. Therefore, in this paper, money is more than cash, namely the issues of the central bank (CB), and also includes banks' issues. Considering that in real life bank-created money takes an even bigger fraction of the money circulated than cash, this feature helps us understand monetary phenomena. Second, the model's economy lasts for a finite number of periods. Still, the CB is able to get its purely nominal issues circulated, contrary to the conventional wisdom. And third, the model is rich enough to accommodate both unconventional quantitative easing policy (QE) and conventional interest-rate policy. Moreover, the mechanism that this paper identifies for the QE to improve efficiency is different to those that the literature has identified, and this mechanism is related to the above-cited difference between the CB-created money and bank-created money, as will be explained later.

The matter of money creation by banks is important in itself. It is typically described in a macroeconomics textbook (when discussing fractional reserve banking) to be carried out in rounds of loan-deposit-loan: a bank first lends out a certain amount cash to someone, who deposits it back to another bank, which then lends it out to someone else. This description seems to suggest that one should track an endless dynamic process in order to consider the money creation by banks. This description, however, may obscure the true nature of that activity: banks create money by issuing a (usually demand deposit) liability, which is essentially a promise to pay (at demand). The above described process is only one way for banks to issue liability, and in that way it is issued to exchange cash. But banks can create money by directly lending their liability to borrowers. Historically, this lending could be done with the issuance of banks' own notes. Nowadays, when most banks do not print notes of their own, it is done by banks putting a number into the borrower's deposit account, with a click of mouse; indeed, usually lending is done in this way when the amount is large – e.g., only rarely do banks disgorge hundred thousands of dollars of greenbacks to a mortgage borrower. Understanding that banks create money by issuing their promises to pay, we come to a simple, static, model to consider their money-creation activity. In the model, entrepreneurs cannot use their promises to pay for hiring workers. Worker, however, accept banks' promises to pay, which entrepreneurs thus borrow in advance of hiring.

To focus on analyzing the money creation by banks in competitive equilibrium, the first part of this paper abstracts away from the CB and assumes bank default to be costless, which allows for the least-fettered private issuance. This paper then introduces the CB in the second part. In doing so it assumes that default is extremely costly to banks. In order to maintain solvency in the negative productivity shock, banks limit the quantity of their issues within a proportion to their wealth. Depending on the aggregate of their wealth, banks may issue too much or too little money. Both situations leave room for the CB to improve efficiency.

The CB, in this paper, is defined as the entity that has two abilities. One is that it can costlessly produce an alternative means of payment which has no intrinsic values, say, shells. The other is that it can enforce repayment from the banks indebted to it. With these two abilities, the CB can get shells circulated in this two-date economy *in one equilibrium*,[1] whereby it can improve over private issuance with certain policy.

Consider first the situation where banks have insufficient wealth and therefore issue

[1] There is always another equilibrium in which no one believes shells to have real value at $t = 1$ and therefore no one accepts them as a means of payment at $t = 0$. There is a similar equilibrium in an infinite-horizon model where no one believes fiat money is accepted as a means of payment and therefore no one accepts it.

an inadequate quantity of money, whereby entrepreneurs get fewer than the first best number of workers. To ease the constraints caused by the money shortage, the CB lends shells to banks – this policy is thus called *quantitative easing* (QE). Specifically, at $t = 0$ the CB lends to all banks a certain quantity of shells and obliges them to pay the same value back at $t = 1$, either with shells or with corn (the real good of the economy), counting 1 kilogram (kg) corn as equivalent to 1kg shells (which gives a nominal value to shells). Then banks lend shells to entrepreneurs and oblige them to pay back at an endogenous interest rate, with either shells or corn, using the same equivalence. With the CB able to enforce banks to repay their debts using that equivalence, shells are circulated in one equilibrium. On the one hand, if workers believe that they can use shells to buy corn at $t = 1$, then they accept their wage to be paid with shells at $t = 0$. On the other hand, if shells have been lent out to entrepreneurs for hiring workers at $t = 0$, shells possess a real value at $t = 1$ because they can be used by the entrepreneurs (and banks) to settle their debts that otherwise have to be settled with corn.

QE improve efficiency in this situation of under-issuance *by substituting inflation for bank default.* Banks cannot issue more because otherwise they would default when the negative productivity shock occurs, invoking a costly procedure of liquidation. By contrast, the CB can keep issuing shells, which causes a reduction in the value of shells, that is *inflation*, in the event of the negative shock,[2] but no liquidation of the CB. This difference arises because of a difference in nature between shells and banks' issues. The CB never commits to redeem its issues with real goods. Shells are thus nominal and there is no need to liquidate the CB in order to find out their value. By contrast, banks commit to redeem their issues at a specified value (i.e. the face value). Thus in case of default, when they cannot pay this specific value, they need to be liquidated to find out the value of their issues. This difference in redeemability gives rise to the above-cited

[2] The low inflations that we see in several economies during and after a long period of quantitative easing policy can be consistent with this model because it predicts inflation only in the event of a severe, negative productivity shock after the QE is implemented.

difference described by Tobin (1963).[3]

This paper finds that at a unique quantity of lending, the QE attains the first best allocation. Moreover, although this policy gives banks the CB's funding at zero interest, it reduces their profit and does not subsidize them (unless their wealth is extremely low). This shows the power of market competition: the CB's funding, exactly because it enlarges *all* banks' lending capacities, subjects them to stronger competition and reduces the profit margin of lending. However, always a single bank strictly prefers getting the CB's funding – thus the CB has no difficulty in implementing the policy – because the enlargement of the bank's capacity *alone* does not reduce the interest rate or the profit margin of lending under perfect competition.

The CB can also implement *interest-rate policy*, by offering to the public an asset that pays interest at the targeted policy rate, with both the principal and interest paid with shells. In the situation where banks issue too much money relative to the first-best allocation, this paper shows that the QE can only make things worse. Interest-rate policy has real effects only if the workers' nominal wage, defined as the total face value of banks' issues that they receive as the wage payment, do not freely adjust with the policy.[4] However, a policy that imposes a tight upper bound on banks' leverage (or a lower bound on their capital adequacy) curbs the over-issuance by banks and improves efficiency. The intuition is that the money created by banks is (part of) their liability and thus its quantity is constrained by the leverage regulation.

Taking it as a premise that banks' liabilities are accepted as a means of payment

[3] Due to the same difference, in real life, depositors run to the banks in hope of getting the full value of their deposits before the banks deplete their liquid assets in the case of bank default, whereas the public do not run to the CB in the case of inflation (unless the CB commits to redeem its currency with bullion or foreign reserves, in which case its currency is not purely nominal any more).

[4] Although a proper examination of this invariance is beyond the scope of this paper, one justification is that entrepreneurs post wage without foreseeing the interest rate policy and certain serious "menu costs" prevent them from changing the wage posts afterwards.

(whereas entrepreneurs' liabilities are not),[5] this paper presents a new approach to central banking in relation to the circulation of nominal claims. Compared to the existing literature on the CB's policy that explicitly models a certain transaction role of money, it has three features (which have been described at the beginning).

First, this paper uses a model of a finite number of periods to consider the circulation of purely nominal issues by the CB, which is not seen in other studies; indeed, the convention wisdom suggests that a infinite horizon model be needed to serve that purpose. Hart and Zingales (2011, 2013) (HZ hereafter), Reis (2009), and Stein (2012) also uses a finite-horizon model to consider certain CB or government policies, but in those papers what government issues must be backed by its tax incomes, and is essentially sovereign debt and not nominal.

Second, the analysis of the CB policy in this paper is built on a competitive-equilibrium analysis of private-sector banks' issuance of money. This is a feature shared with HZ. Also in both HZ and this paper money is means of payment. There is another difference to HZ, besides the one explained above. In their paper, the issuance of money must be collateralized by the real investment and they are interested in the inefficiency of this investment. In contrast, this paper considers uncollateralized bank lending and emphasizes the role of banks' wealth. Brunnermeier and Sannikov (2013) also consider the money creation by financial intermediaries and also underline the role of intermediaries' wealth. However, in their paper money plays the value-storing role,[6] as an alternative to capital, whereas this paper focuses on money's role as a means of payment. Moreover, the two papers are complementary in another respect: their paper

[5] For deep research on this subject, see Kiyotaki and Moore (2001), and a strand of literature that explicitly and strictly models trading and information frictions with a search-matching framework; see Cavalcanti et. al. (1999), Cavalcanti and Wallace (1999), Wallace and Zhu (2007), and Araujo and Minetti (2011). The policy implications of this strand of literature are explored by Williamson (2012, 2013) and Williamson and Wright (2011).

[6] In the papers by Champ, Smith and Williamson (1996) and Freeman (1999), money – including fiat money and banknotes – plays a similar role, as the saving instruments across different locations.

considers the long-term government bonds, while this one considers the QE.

Third, despite its simplicity, this paper's model accommodates both unconventional QE and conventional interest-rate policy. The latter is studied extensively with cash-in-advance (CIA) models (while the cash-less limit assumption of New Keynesian models abstracts from a transaction role of money), for which a survey is provided by Walsh (2010). Moreover, this paper's assumption that entrepreneurs need to borrow money in advance before hiring workers is reminiscent of working-capital models in the manner of Christiano and Eichenbaum (1992, 1995) and Fuerst (1992). However, a fundamental difference to the CIA models consists in the aforementioned second feature of this paper; that is, in those CIA models, money is cash, namely the CB's issues, and banks are financial intermediaries that pass cash from depositors to borrowers,[7] whereas in this paper, money is more than the CB's issues and is also endogenously created by banks on the private sector. Unconventional monetary policy, especially QE, are considered by Gertler and Karadi (2011), Gertler and Kiyotaki (2010) and Reis (2009), among others. A difference is that what the government issues in those papers is backed by tax incomes and thus not nominal (as explained above), whereas in this paper what the CB issues is nominal. As a result, in those papers, QE works by transferring wealth to banks, whereas in this paper, it works by substituting inflation for bank default and may reduce banks' profit.

The rest of the paper is organized as follows. Section 2 sets up the model. The model is analyzed in Section 3 where bank default is assumed to be costless. Section 4, assuming that bank default is prohibitively costly, introduces the CB. Section 5 concludes. Any proofs missing in the main text are relegated to Appendix.

[7] Note that banks' function of money creation considered by this paper is different to that of financial intermediation. In this paper, the real resource that workers "lend" to entrepreneurs is their labor and cannot be deposited; rather, banks issue money to combine workers with entrepreneurs into production.

2 The Model

The economy has one storable good, corn, which is used as the numeraire. There are two dates, $t = 0$ and 1. Contracting and production occur at $t = 0$, yielding and consumption at $t = 1$. There are N banks, N^2 entrepreneurs and N^3 workers, where N is a large number; later, in section 4, we will introduce the central bank. Thus, banks are in perfect competition and each serves a large number of entrepreneurs; and there are more workers than entrepreneurs can hire. All agents are risk-neutral and protected by limited liability.

Workers either produce w kilograms (kg) of corn in autarky, or are hired by entrepreneurs, who each have h units of human (or physical) capital. If an entrepreneur hires L workers at $t = 0$, then his project yields at $t = 1$

$$y = \widetilde{A} h^{1-\alpha} L^{\alpha},$$

where $0 < \alpha < 1$. Without losing any generality, normalize $h = 1$. Productivity, $\widetilde{A}$, is subject to a common shock. At $t = 0$, it is common knowledge that $\widetilde{A} = \overline{A}$ with probability $q > 0$ and $\widetilde{A} = \underline{A}$ with probability $1 - q > 0$. Let $A_e \equiv q\overline{A} + (1-q)\underline{A}$ denote the mean. Assume:

$$0 < \underline{A} < A_e \alpha. \tag{1}$$

As there are more workers than entrepreneurs can hire, entrepreneurs hire workers at a real wage of w, what the workers would earn in autarky.

Banks each have G units of corn, where a unit is defined as N kg and used wherever banks are concerned. If there is no friction, banks are irrelevant to corn production. What makes banks important is the following friction.

Assumption 1 (friction of payment): entrepreneurs cannot use their promises to pay for hiring workers.

One justification for this assumption is that workers cannot enforce entrepreneurs to fulfill their promises to pay at $t = 1$; consequently entrepreneurs face a "borrowing constraint".

Due to this friction of payment, entrepreneurs need to borrow a means of payment that workers accept before hiring, or put differently, they need to have *money in advance.* In this paper, money – namely means of payment – is created by banks, and by the central bank as well, which, however, is not considered until section 4. Banks create money because of the following assumption:

Assumption 2: Workers accept banks' promises to pay as a means of payment.

Back to the discussion of Assumption 1, if it is justified by the borrowing constraint that entrepreneurs face, then Assumption 2 means that workers can enforce banks to fulfill their promises to pay at $t = 1$, although they cannot do that to entrepreneurs.[8]

To enable entrepreneurs to borrow banks' promises to pay, the following is assumed.

Assumption 3: Banks can enforce the indebted entrepreneurs to repay the debts.

It may occur to entrepreneurs, however, that even with all their projects' yields, they still cannot *fully* repay their debts to the creditor banks. In this case, the *entrepreneurs default* and pass all their yields on to the banks.

To fix the idea, suppose that banks' promises to pay are printed on paper, called notes. A note issued by a bank bears the quantity of corn that the bank promises to pay. This quantity is called the note's *face value* or *par value.*

Assumption 4: The face values of banks' notes cannot be contingent on the realization of $\widetilde{A}$.

This assumption captures the real life observation that the repayment (principal and interest) to a demand deposit account (which is the one among banks' liabilities that

[8] Following Kiyotaki and Moore (2001), the two assumptions may be related to a *resale constraint*: suppose there are many, say K, types of goods and each is equally needed for subsistence and produced by N/K entrepreneurs; the workers of an entrepreneur trust his promise to pay them with his product or service, but they cannot use this promise for exchanging other goods, nor can they easily bring his product or service around for that purpose. However, workers can use any bank's promises to pay for buying goods from all the entrepreneurs.

is the most accepted as a means of payment) rarely depends on the state of the real economy. The reason for that is beyond this paper's scope.

Denote by D the aggregate face value of a bank's notes. If a bank does not issue too much, for example, if $D \leq G$, then it can always redeem its notes at face value at $t = 1$, that is, always fulfill its promise to pay F kg corn with this quantity of corn. In this case, the bank's notes are worth their face values, or *valued at par* at $t = 0$. If in some contingency, a bank cannot redeem all its notes at face value, then the *bank defaults* and legal costs of C_d are incurred, which the bank pays before it repays its note holders. Anticipating the possibility of default, at $t = 0$ workers do not take the bank's notes at face value and discount them with a factor of $\delta \leq 1$.

To hire workers, therefore, entrepreneurs borrow notes from banks at $t = 0$. Besides the friction of payment, entrepreneurs face another friction.

Assumption 5 (real friction): entrepreneurs are unable to make commitment on the scale of their projects in terms of the number of workers they hire.

This friction can be regarded as "real," in the sense that it is unrelated to means of payment. Due to this friction, the contract between a bank and an entrepreneur is not conditional on the scale of the latter's project and is as follows. If at $t = 0$, the entrepreneur borrows from the bank its notes of overall face value E, then he owes the bank a debt of $E(1 + r)$, which he repays at $t = 1$ with corn yielded from his project. Thus, r is the interest rate charged by the bank.

The timing is as follows.

At $t = 0$, each bank posts the quantity of its issuance, D, and the interest rate that it charges, r. Both are publicly observed. Each entrepreneur goes to one bank and demands its notes of face value E. If a bank's notes are over-demanded, only a fraction of the entrepreneurs get their demand met. These entrepreneurs then use the borrowed notes to hire workers and start the production.

At $t = 1$, entrepreneurs produce corn and repay $E(1+r)$ to the creditor banks if they can. If they cannot, then they default and give all their output of corn to the banks.

Banks gather corn from the entrepreneurs and from their own storage. If the amount that a bank gathers is no less than its total issues, D, then the bank goes to redeem all its notes at face value. Otherwise, the bank defaults and pays first the legal costs of C_d and then distributes what remains to the holders of its notes in proportion to the face value of their holding. Finally, the agents consume what they have got.

Passing on to the equilibrium analysis, this paper figures out two benchmark allocations.

The First-Best and Second-Best Allocations

Efficiency concerns the number of workers allocated to entrepreneurs. Define the first-best allocation as the number of workers at which the social surplus of each project is maximized. Due to universal risk neutrality and the opportunity cost of labor being w, the social surplus is $A_e L^\alpha - wL$. To maximize it, the first-best allocation is

$$L^{FB} = (\frac{A_e \alpha}{w})^{\frac{1}{1-\alpha}}. \tag{2}$$

The second-best allocation is defined as the number of workers that entrepreneurs would hire in the competitive equilibrium if the friction of payment (in Assumption 1) were absent, but the real friction (in Assumption 5) remained – that is, if entrepreneurs could hire workers with their own promises to pay, but their wage offer could not be contingent on the scale of the projects. The equilibrium allocation is as follows.

Lemma 1 *The second-best number of workers that entrepreneurs hire is:*

$$L^{SB} = (\frac{q\overline{A}\alpha + (1-q)\underline{A}}{w})^{\frac{1}{1-\alpha}}. \tag{3}$$

Obviously, $L^{SB} > L^{FB}$.[9] This wedge between the two allocations allows for a circumstance where banks issue too much money, as will be shown.

[9] This is because of the real friction; if entrepreneurs could compete by posting both wage and scale, then the equilibrium allocation would conform to the first-best. A full-fledged analysis can be found in Wang (2010).

The section below studies banks' issuance of money in circumstances where it is least fettered.

3 The Least Fettered Issuance

In this section, we make one more assumption:

Assumption 6: Default is costless: $C_d = 0$.

This assumption allows for the least fettered issuance: unconcerned about default, banks can choose any scale of issuance, namely, any value of D.

To examine bank issuance in competitive equilibrium, we first consider the demand of banks' notes, then the supply, and, finally, the meeting of the two.

3.1 The Demand Side of the Market for Notes

Consider the entrepreneurs who borrow from a bank that offers (D, r). This profile of (D, r), as will be shown, determines δ, the discount factor of the bank's notes. If an entrepreneur borrows a face value E of the notes, then, given that the notes are worth δE and the real wage is w, he hires L workers, where

$$L = \frac{\delta E}{w}. \tag{4}$$

At $t = 1$, the entrepreneur either repays $E(1 + r)$ of corn to the bank or defaults. Thus, his decision problem is:

$$\max_E q(\overline{A}L^\alpha - E(1 + r)) + (1 - q)\max(\underline{A}L^\alpha - E(1 + r), 0), s.t.(4). \tag{5}$$

Lemma 2 *Entrepreneurs all default in the bad state: For any (w, δ, r), the solution to problem (5) satisfies $\underline{A}L^\alpha < E(1 + r)$.*

Therefore, the entrepreneurs' borrowing maximizes their profit in the good state, $\overline{A}L^\alpha - E(1 + r)$, and, thus, is

$$E(\delta, r) = (\frac{\overline{A}\alpha}{1 + r})^{\frac{1}{1-\alpha}}(\frac{\delta}{w})^{\frac{\alpha}{1-\alpha}}. \tag{6}$$

Then, the amount of labor they hire and their profit are, respectively,

$$L(R) = (\frac{\overline{A}\alpha}{wR})^{\frac{1}{1-\alpha}} \tag{7}$$

$$V(R) = q(1-\alpha)(\frac{\overline{A}^{\frac{1}{\alpha}}\alpha}{wR})^{\frac{\alpha}{1-\alpha}}, \tag{8}$$

where

$$R \equiv \frac{1+r}{\delta}. \tag{9}$$

So defined, R can be regarded as the *real* gross interest rate of borrowing: To obtain a means of payment that is worth 1, an entrepreneur borrows notes of face value $1/\delta$, then in a debt of $(1+r)/\delta$. Naturally, real variables L and V depend only on the real interest, R, and inversely: at a higher real interest rate, the entrepreneurs employ fewer workers and obtain less profit.

Define R^{FB} (R^{SB}) by $L(R) = L^{FB}$ (L^{SB}); that is, at $R = R^{FB}$ (R^{SB}), entrepreneurs hire the first-best (second-best) number of workers. From (2), (3) and (7),

$$R^{FB} = \frac{\overline{A}}{A_e} \tag{10}$$

$$R^{SB} = \frac{\overline{A}\alpha}{q\overline{A}\alpha + (1-q)\underline{A}}. \tag{11}$$

After all banks have posted (D, r), each entrepreneur decides which bank to go to. In the equilibria of this subgame, an entrepreneur gets the same expected profit, $\widehat{V}$, from going to any bank who attracts some entrepreneurs.[10] Define $\widehat{R}$ by $V(\widehat{R}) = \widehat{V}$. Then, $\widehat{R}$ can be regarded as the real interest rate that prevails on the notes market, conditional on the choices of (D, r) by all the banks.

[10] Any single entrepreneur is too small to affect a bank's offering of R and thus takes all banks' offerings as given. No entrepreneur goes to a bank offering $V(R) < \widehat{V}$ when he can get $\widehat{V}$ elsewhere. On the other hand, if a bank offers $V(R) > \widehat{V}$, it induces over-demand for its issues (which is never optimal), so an entrepreneur coming to it is served with such a probability l that $l \cdot V(R) = \widehat{V}$.

3.2 The Supply Side of the Market for Notes

Given there is a large number of banks, a single bank is too small to affect $\widehat{V}$, and thus takes it as given when choosing (D, r). To attract entrepreneurs, a bank offers $V(R) \geq \widehat{V}$ or, equivalently, $R = (1+r)/\delta \leq \widehat{R}$. The interest rate r is chosen by the bank directly. The discount factor, δ, is determined by (D, r) as follows. It depends on whether the bank defaults. In the good state, when the entrepreneurs do not default, the bank has no difficulty redeeming its notes. But in the bad state, by Lemma 2, all the entrepreneurs default, which drags the bank to default if it has issued so much that its wealth, G, is not sufficient to cover the loss, as the proposition below states.

Proposition 1 *If a bank chooses* (D, r)*, then,*

(i) in the bad state, the value of its loans is

$$\underline{Y} = \frac{\underline{A}(1+r)}{\overline{A}\alpha} D, \tag{12}$$

and the bank does not default if and only if

$$D \cdot \left(1 - \frac{\underline{A}(1+r)}{\overline{A}\alpha}\right) \leq G; \tag{13}$$

(ii) the discount factor of the bank's notes at $t = 0$ *is*

$$\delta(D, r) = \left\{ \begin{array}{c} 1, \text{ if (13) is satisfied} \\ q + (1-q)\left(\frac{G}{D} + \frac{\underline{A}(1+r)}{\overline{A}\alpha}\right), \text{ otherwise} \end{array} \right\}. \tag{14}$$

Intuitively, in the bad state, the bank's balance sheet is:

Asset	**Liability**
Corn stored (G)	Equity
Loans to the entrepreneurs ($\underline{Y}$)	Liability to the note holders (D)

Table 1: The balance sheet of a bank in the bad state

It does not default if and only if

$$D \leq G + \underline{Y}, \tag{15}$$

which, with $\underline{Y}$ given by (12), is equivalent to (13).

An intuition for (14) is that a bank's notes are not discounted (i.e., $\delta = 1$) if it will never default; otherwise, its notes are discounted in the bad state at a factor of $(G + \underline{Y})/D$, the quotient of the asset value over the total liability, as $C_d = 0$.

Now consider banks' decision problem at $t = 0$. If a bank chooses (D, r), then in the good state it receives repayment $D(1 + r)$ from the entrepreneurs and pays out D to its note holders. Together with stored corn, G, it obtains $G + Dr$. In the bad state, when the entrepreneurs default, the bank obtains the difference between the asset value and the liability, $G + \underline{Y} - D$, if this difference is non-negative, otherwise the bank defaults and obtains 0 by limited liability. Thus, the net expected profit of the bank, $\Pi(D, r)$, equals $q(G + Dr) + (1 - q)\max(G + \underline{Y} - D, 0) - G$. With $\underline{Y}$ given by (12),

$$\Pi(D, r) = q(G + Dr) + (1 - q)\max(G - (1 - \frac{\underline{A}(1 + r)}{\overline{A}\alpha})D, 0) - G. \tag{16}$$

Taking the real interest rate prevailing on the notes market, $\widehat{R}$, as given, each bank chooses (D, r) to maximize $\Pi(D, r)$, subject to the constraint that it can attract entrepreneurs, that is,

$$\frac{1 + r}{\delta(D, r)} \leq \widehat{R}, \tag{17}$$

where $\delta(D, r)$, the discount factor determined by (D, r), is given by (14). The solution to this problem depends on $\widehat{R}$ and is given below.

Proposition 2 *The solution to and the value of a bank's problem are:*

(i) if $\widehat{R} > R^{SB}$, then $D = \infty$ and $\Pi = \infty$;

(ii) if $\widehat{R} = R^{SB}$, then $\Pi = 0$, and the bank is indifferent to any quantity of issues, D, with r determined by D through the binding (17);

(iii) if $\widehat{R} < R^{SB}$, then lending makes a loss, and thus $D = 0$ and $\Pi = 0$.

By the proposition, the profit margin of lending to entrepreneurs is positive if and only if $\widehat{R} > R^{SB}$. For an intuition, note that a bank wants the interest rate, r, as high as possible, and, thus, constraint (17) is binding. Therefore, all banks offer $R = \widehat{R}$. What a bank obtains by lending to one entrepreneur, $\widehat{\pi}$, is the difference of the social value of his project minus his profit from it; that is, $\widehat{\pi} = A_e L^{\alpha} - wL - V$. With L and V given by (7) and (8) and $R = \widehat{R}$,

$$\widehat{\pi} = (\frac{\overline{A}\alpha}{w})^{\frac{\alpha}{1-\alpha}}(q\overline{A}\alpha + (1-q)\underline{A}) \cdot \widehat{R}^{\frac{-1}{1-\alpha}}(\widehat{R} - R^{SB}). \tag{18}$$

Therefore, the profit margin of lending, $\widehat{\pi}$, is positive if and only if $\widehat{R} > R^{SB}$. If the profit margin of lending is positive, a bank gets $\Pi = \infty$ because it has an infinitely large lending capacity (i.e. $D = \infty$), despite its finite stock of corn. This unlimited capacity is derived from the privilege that banks lend out their liabilities and from the assumption that default is costless.

3.3 The Equilibrium: The Second Best Attained

In equilibrium, defined below, the prevailing real interest rate, $\widehat{R}$, clears the market for notes. Let $\mathbb{N} \equiv \{1, 2, ..., N\}$.

Definition 1 *A profile* $(\{D_i, r_i, \delta_i, \beta_i, E_i\}_{i\in\mathbb{N}}; \widehat{R})$ *forms an equilibrium if:*

(i) given $\widehat{R}$*, it is optimal for bank* i *to choose* (D_i, r_i)*, which determines* $\delta_i = \delta(D_i, r_i)$ *through (14);*

(ii) given banks' choices of $\{D_i, r_i, \delta_i\}_{i\in\mathbb{N}}$*,* $N\beta_i$ *entrepreneurs go to bank* i *and their demand its notes,* E_i*, is optimal:* $E_i = E(\delta_i, r_i)$ *given by (6);*

(iii) the market clears: $D_i = \beta_i E_i$[11] *and* $\sum_{i\in\mathbb{N}} N\beta_i = N^2$.

In any equilibrium, banks neither get an infinitely large profit, nor abstain from lending, which, by Proposition 2, is the case if and only if

$$\widehat{R} = R^{SB}. \tag{19}$$

[11] Note that D is in the unit of N kg, while entrepreneurs' demand is denoted with the unit of kg.

At this value of $\widehat{R}$, entrepreneurs hire L^{SB} workers. Therefore, on the real side, the equilibrium allocation is unique and is the second-best, namely, the one that would arise if the friction of payment were absent. This friction, therefore, is completely overcome by competition between banks under the least fettered issuance.

On the nominal side, however, there is indeterminacy. At $\widehat{R} = R^{SB}$, by Proposition 2, the profit margin of lending is 0, and individual banks are indifferent to any quantity of issues, although in aggregation their issues exactly suffice for entrepreneurs to hire L^{SB} workers. This indeterminacy leads to a continuum of equilibria. In the symmetric one among them, all banks issue the same quantity of notes and their notes are discounted at the same factor, therefore, one bank's notes are perfect substitutes for another's. There are asymmetric equilibria, however, in which, despite ex ante being identical, some banks issue more than others. The former's notes are thus discounted more and lent out at a lower interest rate than the latter's are.

Proposition 3 *(i) In any equilibrium, $\widehat{R} = R^{SB}$, the profit margin of bank issuance is 0, and the second-best allocation is attained.*

(ii) In any equilibrium, a number of banks default at $t = 1$ upon the realization of $\widetilde{A} = \underline{A}$ if and only if banks' aggregate wealth is low enough:

$$G < [(q\overline{A}\alpha + (1-q)\underline{A}) - \underline{A}](\frac{q\overline{A}\alpha + (1-q)\underline{A}}{w})^{\frac{\alpha}{1-\alpha}} \equiv G^{SB}. \tag{20}$$

The profit margin of bank issuance is nullified because banks engage in Bertrand competition with an unlimited capacity, which arises from the privilege that their liabilities are accepted as a means of payment. Therefore, *this privilege actually harms banks under sufficient competition.* Moreover, since banks are indifferent to any quantity of issues, the quantity of money circulated in the economy is determined by the real sectors, namely, the sectors of entrepreneurs and workers. Lastly, the aggregate bank issues are fixed at a quantity exactly sufficient for entrepreneurs to hire the second-best number of workers. If banks' aggregate wealth, G, is small, then it cannot back this scale of notes issuance without causing default.

In this section, by the proposition, banks' issues always draw to entrepreneurs the second-best number of workers. This number is bigger than the first-best one. In this sense, banks always overissue. To engender a situation in which banks under-issue and thus to have a rich setting to consider central banking, in the next section, bank default is assumed as a grave concern to banks.

4 Central Banking in the Two-Date Economy

In this section, instead of assumption 6 (i.e. costless bank default), we assume

Assumption 7: *Bank default is so costly, that is, C_d is so big, that banks disallow it (otherwise their notes would be discounted too much).*

This assumption intends to anchor the lending capacity of a bank (D) to its wealth (G): a bank does not default in the bad state, by Proposition 1(i), if and only if

$$D \cdot (1 - \frac{\underline{A}(1+r)}{\overline{A}\alpha}) \leq G. \tag{21}$$

The same purpose could be served by an assumption related to moral hazard, such as that made by Getler and Kiyotaki (2010), in which the equity value of a bank (i.e., $G + Y - D$; see Table 1) should never fall below αG, with $\alpha \in (0,1)$; otherwise, the bank's owners will abscond with α fraction of its wealth without being caught.

With Assumption 7, in the absence of the central bank (still), the equilibrium is as follows.

Proposition 4 *Suppose bank default is extremely costly and disallowed. (i) If $G \geq G^{SB}$, then in all equilibria the real allocation is the same as in the case of costless bank default: $\widehat{R} = R^{SB}$, the profit margin of bank issuance is 0, and $L = L^{SB}$.*

If $G < G^{SB}$, there is a unique equilibrium in which the profit margin of bank issuance is positive, and $\widehat{R} > R^{SB}$ and is related to G through

$$G = (\frac{\overline{A}\alpha}{w^{\alpha}})^{\frac{1}{1-\alpha}} \frac{1 - \frac{\underline{A}}{\overline{A}\alpha}\widehat{R}}{\widehat{R}^{\frac{1}{1-\alpha}}} \equiv G(\widehat{R}). \tag{22}$$

(ii) If G increases (from 0) to G^{SB}, the real interest rate, $\widehat{R}$, decreases (from $\overline{A}\alpha/\underline{A}$) to R^{SB} and the number of workers hired by entrepreneurs increases to L^{SB}.

If banks' wealth, G, is beyond G^{SB}, by Proposition 3, banks are able to create the second-best quantity of money – which nullifies the profit margin – without falling insolvent. In equilibrium, therefore, the no-default constraint is not binding and the same second-best allocation is attained, as was in the case of costless bank default.

If $G < G^{SB}$, the profit margin of lending is positive because the inadequate wealth of banks limits their lending capacities and thereby relaxes competition. The positive profit margin drives all banks to issue as much as possible, until the non-default constraint, (21), is binding. This clears the indeterminacy in the quantity of issues by individual banks and induces a unique equilibrium. It also shows that, now, the quantity of money circulated is anchored by banks' wealth (G), thus determined by the supply side (i.e., the banking sector), rather than by the demand side, as was in the case of costless bank default. Therefore, the lower the banks' wealth, the less the money they create and, as a result, the higher the interest rate of borrowing ($\widehat{R}$) and the fewer the workers whom entrepreneurs hire.

Given $G(\widehat{R})$ in (22), define $G^{FB} \equiv G(R^{FB})$, the level of wealth at which the real interest rate reaches the first-best value. With R^{FB} given by (10), $G^{FB} = (A_e\alpha - \underline{A})(A_e\alpha/w)^{\frac{\alpha}{1-\alpha}}$. We have $0 < G^{FB} < G^{SB}$.[12] By Proposition 4 (ii), $\widehat{R}$ decreases and L increases with G. Then, relative to the first-best allocation, inefficiency arises in two circumstances.

[12] By the assumption in (1), $A_e\alpha - \underline{A} > 0$. Thus $G^{FB} > 0$. By (20), $G^{SB} = [(q\overline{A}\alpha + (1-q)\underline{A}) - \underline{A}](\frac{q\overline{A}\alpha+(1-q)\underline{A}}{w})^{\frac{\alpha}{1-\alpha}}$. Thus $G^{FB} < G^{SB} \Leftrightarrow A_e\alpha < q\overline{A}\alpha + (1-q)\underline{A} \Leftrightarrow 0 < (1-q)\underline{A}(1-\alpha)$, which holds true.

(1) $G < G^{FB}$. In this circumstance, banks under-issue, whereby the interest rate is too high and entrepreneurs hire too few workers: $\widehat{R} > R^{FB}$ and $L < L^{FB}$.

(2) $G > G^{FB}$. In this circumstance, banks over-issue, whereby the interest rate is too low and entrepreneurs hire too many workers: $\widehat{R} < R^{FB}$ and $L > L^{FB}$.

This paper considers how the central bank in the economy can improve efficiency in both circumstances.

4.1 The Central Bank and Its Policies

The central bank (CB hereafter), in this paper, is defined as the unique entity of the economy that has the following two abilities.

One, it can produce, at zero cost for any quantity, another means of payment which has no intrinsic values; to fix the idea, let it be shells. Shells are different in nature to the money created by bank, i.e., bank notes: the CB does not promise to pay a holder of shells with any real goods, and shells, therefore, are purely nominal, whereas a bank commits to redeem all its notes with the real goods (i.e. corn) before it can take in any profit. As a result, workers accept to be paid with shells at $t = 0$ if and only if they believe that they can use shells to buy corn at $t = 1$. There is always an equilibrium in which they disbelieve so and no shells are circulated at $t = 0$. By contrast, this never occurs to a bank's notes: the bank has G units of corn, therefore its notes always have a positive real value at $t = 1$; indeed, if a bank issues $D \leq G$, workers know, without any equilibrium analysis, that it can always redeem its notes at par. This difference in nature leads to the difference that Tobin (1963) notes, as cited before *Introduction.*

The other ability of the CB in this economy is that it can enforce repayment from banks indebted to it, at any interest rate that it charges.

With this second ability, in one equilibrium, the CB gets shells circulated in this two-date economy, whereby it can implement two types of policy as follows; the CB announces its policy at $t = 0$, before banks post (D, r).

The first type is *interest-rate policy*, under which the CB sets the risk-free interest rate at $r_p \geq 0$. To do that, the CB offers a savings account to the agents of the economy. If the CB receives a deposit of some banks' notes of overall face value F kg of corn at $t = 0$, it issues to the depositors $F(1 + r_p)$ kg shells at $t = 1$. By taking in the notes, the CB becomes a creditor to the issuing banks and charges them an interest rate of $1 + r_p + \varepsilon$ for some $\varepsilon > 0$. Thus it obliges these banks altogether to repay $F(1 + r_p + \varepsilon)$, either with corn or with shells, counting 1kg corn equivalent to 1kg shells. With this equivalence, we call 1kg corn as the *par value* of 1kg shells. As noted above, there is an equilibrium in which no one believes that shells have real value at $t = 1$ and thus no one deposits any bank notes with the CB at $t = 0$, which makes the policy meaningless. However, as the CB is able to impose that interest rate on the indebted banks and to enforce repayment from them, there is an equilibrium in which the policy is meaningful. In this equilibrium, 1kg of shells is worth 1kg corn, its par value, at $t = 1$. On the one hand, shells can never be priced above that, otherwise the banks indebted to the CB will not use shells, but only corn, to settle their debts – thus shells are worth nothing rather than above the par. On the other hand, shells are not priced under the par either. Otherwise the indebted banks would want to use only shells to settle all their debts. Thus, their demand for shells would be $F(1 + r_p + \varepsilon)$ kg, but only $F(1 + r_p)$ kg shells are issued, below the demand: not in equilibrium.[13]

However, banks can escape the punitive interest rate $1 + r_p + \varepsilon$ by paying their note holders the same interest rate of $1 + r_p$ (or even a little more), thereby keeping them to stay. As a result of the policy rate set at r_p, therefore, a bank's note of face value 1 issued at $t = 0$ is worth $1 + r_p$ at $t = 1$, and banks, if issuing notes of aggregate face value D at $t = 0$, are in a liability of $D(1 + r_p)$ at $t = 1$.

[13] This $\varepsilon > 0$ thus helps pin down a unique price for shells by inducing over-demand for shells. In its absence, namely, if $\varepsilon = 0$, the shell price could be anywhere in $(0, 1]$: the holders of shells are willing to sell all $F(1 + r_p)$ of shells at any price $p > 0$ and the indebted banks want to buy exactly this amount at any price $p \leq 1$.

The second type of policy is *quantitative easing policy*, whereby the CB lends shells to all banks in order to enlarge the money supply. As the CB can enforce repayment from banks, shells receive a real value at $t = 1$ and are therefore circulated at $t = 0$ in one equilibrium, as is shown in the next subsection, where we demonstrate that this policy is the optimal response of the CB to the inadequate creation of money by banks.

4.2 Quantitative Easing Policy in the Case of $G < G^{FB}$

In this case, endowed with low wealth and concerned about default, banks under-issue relative to the first best allocation, causing a credit crunch, with symptoms that the interest rate is too high (i.e., $\widehat{R} > R^{FB}$) and that the scale of entrepreneurs' projects is too small (i.e., $L < L^{FB}$). To ease the constraint imposed by the shortage of money, the CB implements a quantitative easing policy. Specifically, at $t = 0$ the CB lends to all N banks S units of shells each (again 1 unit defined as N kg) and obliges them to pay the same value back at $t = 1$, either with shells or with corn, counting 1kg corn equivalent to 1kg shells. Then banks have two types of money to lend to entrepreneurs. One is their own notes, the other is shells. We assume that an entrepreneur borrows only shells or only a bank's notes, but not both of them, in order to avoid the problem of which debt among the two is senior.

A loan contract for banks' notes is as before: If an entrepreneur borrows from a bank its notes of face value E, then he owes the bank a debt of $E(1 + r)$, which he repays at $t = 1$ with corn.

A loan contract for shells is as follows. If an entrepreneur borrows E_s kg shells, then he owes the bank a debt of $E_s(1 + r_s)$, which he repays at $t = 1$ with either corn or shells, counting 1kg corn equivalent to 1kg shells. Consider, first, the case in which S is not too big so that $r_s > 0$ in equilibrium; by Proposition 5 below, that requires $S \leq \frac{\overline{A}\alpha}{q(\overline{A}\alpha - \underline{A})}(G^{SB} - G)$, which holds for the optimal value of S. In this case, all the shells are lent out and the aggregate debt of shell-borrowing entrepreneurs $NS \times (1 + r_s)$.

To simplify exposition, we assume that even in case of default, the shell-borrowing entrepreneurs still want to pay out as much of their debt as possible; therefore, if they repay their debts with shells, they try to buy as close to $E_s(1+r_s)$ kg shells as possible.

As noted above, no one commits to redeem shells with corn at $t = 1$. At $t = 0$ workers accept their wage to be paid with shells if and only if they believe that they can use shells to buy corn at $t = 1$. And there is an equilibrium in which they disbelieve so and thus shells are not circulated. But we are interested in another equilibrium in which shells are circulated: workers accept payment with shells at $t = 0$ – as a result, NS units of shells are in their hands; and indeed shells receive a positive real value at $t = 1$. They have a real value because the shell-borrowing entrepreneurs want to use them to settle their debts to the banks, and banks to the CB. Thus, at $t = 1$ the timing of events is now as follows.

First, entrepreneurs' projects yield corn. Second, the market for shells opens, on which the shell-borrowing entrepreneurs buy shells from workers; let p_1 denote the shell price on this market. Third, entrepreneurs settle their debts to the creditor banks: the notes borrowing entrepreneurs use corn and the shell-borrowing ones use corn and/or shells. Fourth, the market for shells may open the second time, on which the banks having more than S units of shells sell their excesses to the banks short of S units of shells. Note that the aggregate excess exactly equals the aggregate shortage. Finally, banks redeem their notes from workers and settle their debts to the CB (and then consumption starts).

Assumption 8: If the CB ends up with a certain amount of corn, it will use all this corn to buy the shells remained with banks.

This is assumed to pin down the price of shells on the second shell market (between banks), as is made clear below.

Consider p_1, the price of shells at $t = 1$ on the first market where entrepreneurs buy shells from workers. In the good state when $\widetilde{A} = \overline{A}$ and no entrepreneurs default, $p_1 = 1$, as follows. On the one hand, $p_1 \leq 1$, otherwise, no shells, but corn, would be used by the shell-borrowing entrepreneurs to settle their debts, and thus $p_1 = 0$ instead of $p_1 > 1$. On the other hand, in this state, p_1 cannot go below 1 either. Suppose, otherwise, $p_1 < 1$. Then, these entrepreneurs would want only shells (and not corn) to repay all their debts. Their demand for shells would thus be $NS(1 + r_s)$ (as they do not default), which, with $r_s > 0$, is bigger than NS, the supply of shells: *not in equilibrium*. As $p_1 = 1$, the shell-borrowing entrepreneurs are indifferent to repaying their banks with shells or with corn. As a result, some banks may end up with more than S units of shells, some less. The former wants to sell shells to the latter. Thus, the second market for shells opens. In the absence of Assumption 8, any price in $(0, 1]$ can clear this market: the supply side wants to sell all the excessive shells at any positive price, while the demand side is willing to buy exactly the amount supplied at any price $p \leq 1$. The assumption, however, pins down the price to be 1: if a bank holds on selling his Q units of excessive shells, then some other banks must be short of shells of the same amount to repay the CB, and they have to make it up with Q units of corn, which the CB, by Assumption 8, pays to the bank to buy his excessive Q units of shells; thus no banks sell their excessive shells at price $p < 1$.[14]

In the bad state when $\widetilde{A} = \underline{A}$, entrepreneurs all default. The shell-borrowing entrepreneurs use all their yields to buy shells, as is assumed. As banks and entrepreneurs are symmetric, the shell-borrowing entrepreneurs of one bank altogether buy back $1/N$ of the shells on the market, which is S units of shells. Namely, each bank ends up with S units of shell, exactly what it needs to repay the CB. Thus, the second market for shells does not open in this state – so Assumption 8 plays no role. Let Y_s denote the

[14] Another way to pin down the shell price on the second market is to let the CB charge a tiny interest rate in its lending to banks, which would play the same role as ε does in the case of interest rate policy (see the preceding footnote). This way, however, introduces an unnecessary (but still tractable) technical complication, that the no-default constraint, (21), need to be changed to take care of this interest payment to the CB.

total yields of the shell-borrowing entrepreneurs of one bank. Then, the shell price in the bad state is $p_1 = Y_s/S$. With a calculation similar to that leading to equation (12), $Y_s/S = \underline{A}(1 + r_s)/(\overline{A}\alpha)$, and, as is shown below in Proposition 5, $\underline{A}(1 + r_s)/(\overline{A}\alpha) < 1$. Thus, in the bad state, $p_1 = \underline{A}(1 + r_s)/(\overline{A}\alpha) < 1$.

At $t = 0$, the real value of 1kg shells to workers, denoted by, δ_s, equals the mean of these two state-contingent prices at $t = 1$. Thus,

$$\delta_s = q \cdot 1 + (1 - q) \cdot \frac{\underline{A}(1 + r_s)}{\overline{A}\alpha}.^{15} \tag{23}$$

As $\underline{A}(1 + r_s)/(\overline{A}\alpha) < 1$, we have $\delta_s < 1$.

As $p_1 < 1$ in the bad state, shells are worth less in the bad state than they are in the good state. This decrease in the real value of the CB's issues means inflation. The inflation is what enables the quantitative easing policy to improve efficiency over the private-sector issuance, because it makes the real value of the CB's issues contingent on the realization of the productivity shock, $\widetilde{A}$. Considering that banks' wealth is below G^{FB}, to make entrepreneurs' projects reach the first best scale (i.e. L^{FB}) entails that what workers as a whole receive varies with the productivity shock. In the absence of the CB's intervention, the real value of banks' notes is made contingent on the productivity shock only through costly default by Assumption 4. By contrast, the real value of the CB's issues is made contingent through inflation. Therefore, inflation is an alternative of bank default to make the value of money contingent on the productivity shock. But unlike bank default, inflation does not trigger costly procedures of winding down banks. This difference arises because the CB never commits to redeem its issues with real goods, whereas private banks commit to redeem their issues at the specified value (i.e. the face value). As a result, in the case of default, banks need to be liquidated in order to find out the value of their liabilities, whereas in the case of inflation, there is no need to liquidate the CB to see the value of shells.

[15] Alternatively, this equation can be derived from equation (14) by replacing G with 0 (and replacing r with r_s), because shells are backed not by any banks' wealth, but only by the loans.

With the quantitative-easing policy, the equilibrium is as follows.

Proposition 5 *(i) If the central bank lends to each bank* $S \leq \underline{S}(G) \equiv \frac{\overline{A}\alpha}{q(\overline{A}\alpha - \underline{A})}(G^{SB} - G)$ *units of shells, then there is a unique equilibrium in which the real interest rate,* $\widehat{R}$*, is determined by* S *through*

$$S = (\frac{\underline{A}}{w^{\alpha}})^{\frac{1}{1-\alpha}} \frac{1-(1-q)\theta}{q\theta^{\frac{1}{1-\alpha}}} - \frac{1-(1-q)\theta}{q(1-\theta)} G \equiv F(\theta), \text{ with } \theta \equiv \frac{\underline{A}}{\overline{A}\alpha}\widehat{R}, \tag{24}$$

and the interest rate of lending shells is

$$r_s = \frac{[q\overline{A}\alpha + (1-q)\underline{A}]\widehat{R} - \overline{A}\alpha}{\overline{A}\alpha - (1-q)\underline{A}\widehat{R}}. \tag{25}$$

It satisfies $r_s > 0$ *and* $\underline{A}(1 + r_s)/(\overline{A}\alpha) < 1$.

(ii) If S *increases,* $\widehat{R}$ *and* r_s *decrease and the number of workers hired by entrepreneurs,* L*, increases. At* $S = \underline{S}(G)$, $\widehat{R} = R^{SB}$, $r_s = 0$, *and* $L = L^{SB}$.

(iii) If $G \geq \underline{G} \equiv G(\frac{1}{\alpha}R^{SB})$*, where function* $G(\widehat{R})$ *is given in (22), banks' profit decreases with* S *always. If* $G < \underline{G}$*, then* $F(\frac{\underline{A}}{\overline{A}\alpha} \cdot \frac{1}{\alpha}R^{SB}) > 0$ *and banks' profit is increasing with* S *for* $S \leq F(\frac{\underline{A}}{\overline{A}\alpha} \cdot \frac{1}{\alpha}R^{SB})$ *and decreasing otherwise.*

(iv) The optimal quantity of the CB's lending is $S = F(\frac{\underline{A}}{\overline{A}\alpha} \cdot R^{FB})$*, within* $(0, \underline{S}(G))$*, whereby the first-best allocation is attained (*$L = L^{FB}$*).*

Some comments and intuitions are as follows.

First, *a quantitative-easing policy affects the real interest rate,* $\widehat{R}$*, thus efficiency, if and only if* $\widehat{R}$ *is above threshold* R^{SB}. The "if" part is given by result (ii), which says that if $\widehat{R} > R^{SB}$, the (further) issuance of shells (i.e., a bigger S) lowers $\widehat{R}$. As for the "only if" part, note that by Proposition 2 (iii), by no means can $\widehat{R}$ be dragged down below R^{SB}, otherwise banks would stop lending altogether, not a case in equilibrium. At $\widehat{R} = R^{SB}$, any further issuance by the CB – namely, $S > \underline{S}(G)$ – will not increase the quantity of money circulated, but crowd out the bank-created money or stay in the vaults of banks.

Second, to understand result (iii), note that a quantitative easing policy is both good news and bad news to banks. The good news is that it enlarges their lending capacities. The bad news is that with their capacities all enlarged, they face stronger competition, which reduces the profit margin of lending. By the result, the negative side always dominates unless banks' aggregate wealth is very small, i.e., $G < \underline{G}$. Therefore*, under perfect competition, a quantitative easing policy, though giving banks free funding of the CB, does not subsidize the banks, but reduces their profit, unless* $G < \underline{G}$. Even when the banking sector as whole is worse off from the free funding of the CB, however, an individual bank strictly prefers demanding it if $r_s > 0$ – thus the CB does not need to force banks to accept its loans – because in perfect competition the increase in the capacity of any single bank alone is too insignificant to reduce the profit margin of lending.

Third, the first-best is achieved with the optimal monetary policy, which, therefore, helps not only with the nominal friction (as in Assumption 1), but also with the real friction (as in Assumption 5).

4.3 Interest Rate Policy (or Leverage Regulation) in the Case of $G > G^{FB}$

In this case, banks issue too much money, making the real interest rate too low and entrepreneurs hire too many workers relative to the first-best allocation. Quantitative-easing policy, by Proposition 5, either is devoid of real effects (if $\widehat{R} = R^{SB}$), or lowers the real interest rate further, making things worse. Interest-rate policy, as shown below, has real effects and can implement the first-best allocation if and only if the nominal wage to workers, defined as the total face value of banks' notes that they receive as the wage payment, remains invariant to the policy.[16] Lastly, we show that in this circumstance

[16] Although a proper examination of this invariance is beyond the scope of this paper, one way to accommodate it in the model is to assume that entrepreneurs first post wage offers anticipating no

of over-issuance, proper leverage regulation alone attains the first-best allocation.

Suppose that the CB sets the policy rate at r_p. As a result, a note of face value 1 issued at $t = 0$ is worth $1 + r_p$ at $t = 1$, and banks, if issuing D at $t = 0$, face a liability of $D(1+r_p)$ at $t = 1$. With this amount of liability and the bad-state value of the loans, $\underline{Y}$, given by (12), banks stay solvent in the bad state if and only if

$$D(1 + r_p) \leq G + \frac{\underline{A}(1 + r)}{\overline{A}\alpha} D. \tag{26}$$

The analysis that follows depends on whether the nominal wage at $t = 0$ adjusts or remains invariant with policy rate r_p. Let us start with the case in which the nominal wage adjusts. In this case, at $t = 0$ workers accept a nominal wage of $w/(1 + r_p)$. By borrowing E, an entrepreneur hires $E(1 + r_p)/w$ workers. It follows that with a bank's issues of aggregate face value D, $L = D(1+r_p)/w$ workers are hired and that the discount factor in (4) is now $\delta = 1 + r_p$, which, with (9), implies $R = (1 + r)/(1 + r_p)$. Substitute $R(1 + r_p)$ for $1 + r$ and wL for $D(1 + r_p)$ in (26) and note that the number of workers hired is $L = (\overline{A}\alpha/wR)^{\frac{1}{1-\alpha}}$ by (7). Then, from (26) it follows:

$$w(\frac{\overline{A}\alpha}{wR})^{\frac{1}{1-\alpha}}(1 - \frac{\underline{A}R}{\overline{A}\alpha}) \leq G. \tag{27}$$

As all banks offer $R = \widehat{R}$, this no-default constraint, when binding, is equivalent to (22) in Proposition 4. By Proposition 4(ii), the real interest determined by (22) is above R^{SB} if and only if $G < G^{SB}$. Moreover, if and only if $R > R^{SB}$, by Proposition 2, issuance bears a positive profit margin, which drives banks to issue as much as possible until the no-default constraint, (27), is binding. It follows that in equilibrium, if and only if $G < G^{SB}$, constraint (27) is binding and the real interest is the same as given by (22); and if $G \geq G^{SB}$, the real interest rate is R^{SB} (because it can never fall below R^{SB} by Proposition 2). In both cases, the real interest rate, which determines all the other real variables, is the same as given by Proposition 4 – as it is in the absence of the CB. Therefore,

policy intervention and then are taken with surprise by the interest rate policy, and they are prevented by certain serious "menu costs" from changing the wage posts afterwards.

Proposition 6 *If workers' nominal wage adjusts with the policy rate, the interest-rate policy has no real effects, but deflates the nominal wage to* $w/(1+r_p)$.

Now, consider the case where workers do not accept a nominal wage below w, due to some reason beyond the scope of this paper.[17] Then, borrowing notes of face value E, an entrepreneurs still hires E/w (rather than $E(1+r_p)/w$) workers. This implies that the number of workers hired with a bank's issues of aggregate face value D is $L = D/w$ and that the discount factor in (4) is $\delta = 1$, whereby the real interest rate is $R = 1 + r$. Substitute R for $1 + r$ and wL (with $L = (\overline{A}\alpha/wR)^{\frac{1}{1-\alpha}}$ by 7) for D in (26), which then becomes:

$$w(\frac{\overline{A}\alpha}{wR})^{\frac{1}{1-\alpha}}[(1+r_p) - \frac{\underline{A}}{\overline{A}\alpha}R) \leq G. \tag{28}$$

This no-default constraint is binding if the profit margin of bank issuance is positive, as before. Given that the workers' real wage is now $w(1+r_p)$, the profit to a bank from lending to one entrepreneur becomes $A_e L^\alpha - w(1+r_p)L - V$. With L and V as functions of R given by (7) and (8), it equals $[\overline{A}\alpha/(w^\alpha R)]^{\frac{1}{1-\alpha}}[R/R^{SB} - (1+r_p)]$.This profit margin never goes below 0. Therefore,

$$R \geq R^{SB}(1+r_p). \tag{29}$$

The equilibrium real interest rate is pinned down by conditions (28), (29), and that if (29) is not binding – that is, if the profit margin of issuance is positive – then the no-default constraint, (28), is binding. The results are summarized as follows.

Proposition 7 *Suppose that workers' nominal wage remains invariant to the interest rate policy. (i) With the policy rate set at* r_p*, the equilibrium real interest rate of bank*

[17] Introducing this constraint regarding the nominal wage would not affect the analysis of quantitative-easing policy in Subsection 4.2, where workers are paid either with banks' notes of overall par value w or with the CB notes of overall par value $w/\delta_s > w$.

lending, R, is as follows. For $G < G^{SB}$, R is the root of

$$1 + r_p = \frac{Gw^{\frac{\alpha}{1-\alpha}}}{(\overline{A}\alpha)^{\frac{1}{1-\alpha}}} R^{\frac{1}{1-\alpha}} + \frac{\underline{A}}{\overline{A}\alpha} R \equiv \Phi(R) \tag{30}$$

if $r_p \leq r_p^*$, otherwise $R = R^{SB}(1 + r_p)$, where

$$r_p^* \equiv \frac{[q(\overline{A}\alpha - \underline{A})/G]^{\frac{\alpha}{1-\alpha}}(q\overline{A}\alpha + (1-q)\underline{A})}{w} - 1,$$

and $r_p^* > 0$ if $G < G^{SB}$. For $G \geq G^{SB}$, $R = R^{SB}(1 + r_p)$ for all $r_p \geq 0$.

(ii) The optimal policy rate, with which the first-best allocation is attained, is:

$$r_p^{FB} = \left\{ \begin{array}{c} \Phi(R^{FB}) - 1 \text{ if } G \leq G_s \\ R^{FB}/R^{SB} - 1 \text{ if } G > G_s \end{array} \right\},$$

where $G_s \equiv [(q\overline{A}\alpha + (1-q)\underline{A}) - \underline{A}](A_e\alpha/w)^{\frac{\alpha}{1-\alpha}}$ and satisfies $G^{FB} < G_s < G^{SB}$.

Intuitively, in the case of invariant nominal wages, a higher policy rate raises the real wage of workers and thereby reduces real economic activity, measured by the scale of entrepreneurs' projects. Interestingly, *a high policy rate hurts banks more*: if it is above r_p^*, while entrepreneurs still earn a positive profit, banks get 0 profit, as the constraint of non-negative profit margin, (29), becomes binding. This once again shows the power of competition: a high policy rate diminishes entrepreneurs' demand for bank-created money, thus subjecting banks to stronger competition.

By Proposition 6, if the nominal wage of workers adjusts with the policy rate, interest rate policy is devoid of real effects in this economy. In this situation, however, the CB can still help attain the first-best allocation if it has another authority which has not yet been introduced, that is the authority to regulate banks' leverage rate. The aggregate face value of money that is needed for entrepreneurs to hire the first-best number of workers is $wL^{FB} := D^{FB}$. Suppose the CB imposes the constraint that banks cannot issue more than D^{FB}/G times of their wealth, G. Then, this constraint is binding in the competitive equilibrium if $G \geq G^{FB}$ because by Proposition 4, in its absence, banks issue more than D^{FB} (as $\widehat{R} < R^{FB}$) and thus violate the constraint. With the constraint, therefore, banks issue D^{FB} and the first-best allocation is attained. One way to implement this constraint is the familiar regulation on banks' leverage (i.e. debt-to-equity ratio), $D/(G+Y-D)$, where Y is the value of banks' loans measured at $t=0$. Define

$$l^{FB} := \frac{D^{FB}}{G + \frac{(1-q)\underline{A}(1-\alpha)}{A_e \alpha} D^{FB}},$$

which we show in the proof of the proposition below is the leverage of banks in the first-best allocation.

Proposition 8 *If the CB imposes regulation that disallows the debt to equity ratio of a bank to go above l^{FB}, then banks are subject to the constraint that $D/G \leq D^{FB}/G$ and the first-best allocation is attained.*

However, note that in the circumstances of $G < G^{FB}$ (i.e. of bank under-issuance), this leverage regulation will partly offset the effects of the quantitative easing policy. This is because due to the discount of the CB's issues (i.e. $\delta_s < 1$), the sum of the optimal CB's lending (as given by Proposition 5.iv) plus banks' issues is greater than D^{FB}. *In the circumstances where the CB is implementing a quantitative easing policy, therefore, the leverage policy should be made more lenient than it is in the circumstances of bank over-issuance.*

5 Conclusion

Based on a general equilibrium analysis of private banks' issuing their liabilities as a means of payment, this paper considers the principles of central banking in a two-date economy. Although the economy lasts only for a finite number of periods, the central bank gets its purely nominal issues circulated. This paper shows that in circumstances where banks, constrained by their low wealth, issue an inadequate quantity of money, a quantitative easing policy improves efficiency. Moreover, it works by substituting inflation for bank default as the way to make the real value of money contingent on the state of the real economy. Inflation offers a cheaper way than bank default to serve that purpose because of a difference in nature between bank-created money and the currency (i.e. money created by the central bank): the former is real and its value is backed by the banks' assets, whereas the latter is nominal and its value is not backed by (though related to) the central bank's assets. This paper also studies the circumstances in which banks issue too much money (due to some real friction) and finds that a high policy rate curbs this over-issuance only if workers' nominal wage cannot freely adjust with the policy rate, where the nominal wage is defined as the overall face value of banks' notes that workers receive as the wage payment. However, a regulation that puts an upper bound on banks' leverage rate (or a lower bound on their capital adequacy rate) always works to curb that over-issuance. This paper's model is quite stylized in many respects and unfit for quantitative exercises, but it sheds new insights in an easily understood way on certain important issues, such as quantitative easing policy, interest rate policy, bank capital adequacy regulation, and inflation.

Appendix: The Proofs

Of Lemma 1:

In equilibrium, only one promised wage, denoted by F, prevails on the market, as will be shown. Competitive equilibrium is thus defined as a profile of (F, L), such that:

(a) Given that F prevails on the market, the optimal labor demand of each entrepreneur is L;

(b) Given that each entrepreneur demands L workers, F clears the labor market.

The two conditions are elaborated as follows.

For (a): Given F, each entrepreneur's decision problem on labor demand is:

$$\max_L q(\overline{A}L^\alpha - FL) + (1-q)\max(\underline{A}L^\alpha - FL, 0),$$

where the "max" term appears because the entrepreneur might default in the bad state. That is indeed the case at the optimum. Otherwise, the entrepreneur's problem is

$$\max_L q(\overline{A}L^\alpha - FL) + (1-q)(\underline{A}L^\alpha - FL).$$

The solution is $L = (\frac{A_e\alpha}{F})^{\frac{1}{1-\alpha}}$. Then in the bad state his output is $\underline{A}((\frac{A_e\alpha}{F})^{\frac{\alpha}{1-\alpha}}$, which is smaller than $F \cdot (\frac{A_e\alpha}{F})^{\frac{1}{1-\alpha}}$, the wage obligation, because $\underline{A} < A_e\alpha$ as assumed in (1). Hence, he defaults in the bad state, contradictory to what was supposed.

Defaulting in the bad state, entrepreneurs choose L to maximize the profit in the good state, $\overline{A}L^\alpha - FL$. Therefore, given F, the labor demand is:

$$L = (\frac{\overline{A}\alpha}{F})^{\frac{1}{1-\alpha}}. \tag{31}$$

For (b): As there are a lot more workers than entrepreneurs can hire, the labor market is cleared by an expected wage income of w, the output of workers in autarky. In the good state, the workers hired get the promised wage, F. In the bad state, entrepreneurs default and all the output goes to the workers, each obtaining $\frac{\underline{A}L^\alpha}{L} = \underline{A}L^{\alpha-1}$. The expected wage equals w, and thus the labor market clears, if

$$qF + (1-q)\underline{A}L^{\alpha-1} = w. \tag{32}$$

Equations (31) and (32) together give (3).

Now, show that only one F prevails on the market. If an entrepreneur posts F, then by (31) he hires $L = (\frac{\overline{A}\alpha}{F})^{\frac{1}{1-\alpha}}$ workers, whose wage income is F in the good state and $\underline{A}L^{\alpha-1} = \frac{\underline{A}}{\overline{A}\alpha}F$ in the bad state. Both increase with F. Therefore, workers go only to entrepreneurs who post the highest F, and in equilibrium, only one F prevails.

Q.E.D.

Of Lemma 2:

Suppose, otherwise, an entrepreneur does not default in the bad state. Then, his problem is:

$$\max_{E} q(\overline{A}L^{\alpha} - E(1+r)) + (1-q)(\underline{A}L^{\alpha} - E(1+r)) \text{ s.t. } (4).$$

From the constraint, $E = wL/\delta$. Substitute it into the objective and let $\gamma \equiv w(1+r)/\delta$. Then, the problem becomes

$$\max_{L} A_e L^{\alpha} - \gamma L.$$

The solution is $L = (A_e\alpha/\gamma)^{\frac{1}{1-\alpha}}$. At this scale, the entrepreneur will default in the bad state: $\underline{A}L^{\alpha} < E(1+r)|_{E=wL/\delta;\gamma\equiv w(1+r)/\delta} \Leftrightarrow \underline{A}L^{\alpha} < \gamma L \Leftrightarrow \underline{A}L^{\alpha-1} < \gamma|_{L=(A_e\alpha/\gamma)^{\frac{1}{1-\alpha}}} \Leftrightarrow$ $\underline{A} < A_e\alpha$, which is assumed in (1) – hence a contraction to what was supposed.

Q.E.D.

Of Proposition 1:

(i): As the entrepreneurs all default in the bad state and each hands over his whole output, $\underline{y}$, to the bank, the value of the bank's loans in the bad state, $\underline{Y}$, equals $\underline{y}$ times the number of entrepreneurs that the bank finances, D/E. With $\underline{y} = \underline{A}L^{\alpha}$, and E and L given by (6) and (7),

$$\frac{\underline{y}}{E} = \frac{\underline{A}(1+r)}{\overline{A}\alpha}. \tag{33}$$

Then, $\underline{Y} = \underline{y} \cdot D/E = D \cdot \underline{y}/E = \underline{A}(1+r)/\overline{A}\alpha \cdot D$, that is (12). With $\underline{Y}$ so given, the condition of no-default in the bad state, $D \leq \underline{Y} + G$, becomes (13).

(ii): By (i), if condition (13) holds true, the bank never defaults and is always able to redeem its notes at par at $t = 1$. Therefore, its notes are valued at par at $t = 0$, that is, $\delta = 1$, which gives rise to the upper branch of (14).

If condition (13) is violated, at $t = 1$, the bank defaults in the bad state. All the value of its assets, $G + \underline{Y}$, goes pro rata to the note holders and a note of face value 1 is thus worth $(G + \underline{Y})/D$, which, with $\underline{Y}$ given by (12) (note that the derivation of (12) is independent of whether the bank is solvent or not), equals

$$\frac{G}{D} + \frac{\underline{A}(1+r)}{\overline{A}\alpha}.$$

At $t=1$ in the good state, when the entrepreneurs do not default and, thus, neither does the bank, the notes are worth their par values. At $t=0$, the value of a note of face value 1, namely, the discount factor of the notes, equals the mean of its value at $t=1$ and is as given by the lower branch of (14).

Q.E.D.

Of Proposition 2:

With rearrangement, the profit of a bank choosing (D,r) is:

$$\Pi(D,r)=\left\{\begin{array}{c}\frac{q\overline{A}\alpha+(1-q)\underline{A}}{\overline{A}\alpha}(r-r^{SB})D,\ \text{if } D(1-\frac{\underline{A}(1+r)}{\overline{A}\alpha})\le G\\ qDr-(1-q)G,\ \text{otherwise}\end{array}\right\}, \tag{34}$$

where

$$r^{SB}\equiv\frac{(1-q)(\overline{A}\alpha-\underline{A})}{q\overline{A}\alpha+(1-q)\underline{A}}=R^{SB}-1. \tag{35}$$

Given D, banks want r as high as possible. Hence, (17) is binding, which implies

$$r=\widehat{R}\cdot\delta(D,r)-1. \tag{36}$$

We prove the Proposition by proving the following lemmas.

Lemma 3A: if $\widehat{R}\ge\frac{\overline{A}\alpha}{\underline{A}}$, then $D=\infty$, $r=\widehat{R}-1$, and $\Pi=\infty$.

Proof. In this case, a bank chooses $D=\infty$, $r=\widehat{R}-1$, without default – thus $\delta=1$ – because $1+r=\widehat{R}\ge\frac{\overline{A}\alpha}{\underline{A}}$ and, thus, the condition for non-default, (13), is honored for any D; intuitively, at such a high $\widehat{R}$, the bank issue can be solely and fully backed by the loans. As the bank never defaults, its profit is given by the upper branch of (34). The profit margin $r-r^{SB}=\widehat{R}-R^{SB}>0$, which with $D=\infty$ implies $\Pi=\infty$. ■

For the remaining cases in which $\overline{A}\alpha/\underline{A}>\widehat{R}$ and thus $1-(1-q)\underline{A}/(\overline{A}\alpha)\cdot\widehat{R}>0$, note that if $D(1-\frac{\underline{A}(1+r)}{\overline{A}\alpha})>G$ – namely, if a bank chooses to default – then solved from (36), with $\delta(D,r)$ given by (14):

$$r=\frac{qD+(1-q)G}{D[1-(1-q)\underline{A}/(\overline{A}\alpha)\cdot\widehat{R}]}\widehat{R}-1. \tag{37}$$

Substitute it into the lower branch of (34), then the profit of a bank who chooses to default is

$$\frac{(\widehat{R}/R^{SB}-1)}{\overline{A}\alpha-(1-q)\underline{A}\widehat{R}}[qD+(1-q)G]. \tag{38}$$

Lemma 3B: if $\frac{\overline{A}\alpha}{\underline{A}} > \widehat{R} > \frac{\overline{A}\alpha}{q\overline{A}\alpha+(1-q)\underline{A}}(= R^{SB})$, then $D=\infty$, $r=\frac{(q\overline{A}\alpha+(1-q)\underline{A})\widehat{R}-\overline{A}\alpha}{\overline{A}\alpha-(1-q)\underline{A}\widehat{R}}$, and $\Pi=\infty$.

Proof. In this case, by (36) and $\delta \leq 1$, $1+r<\widehat{R}<\frac{\overline{A}\alpha}{\underline{A}}$. Thus, $1-\frac{\underline{A}(1+r)}{\overline{A}\alpha}>0$. The non-default condition, (13), is equivalent to $D \leq G(1-\frac{\underline{A}(1+r)}{\overline{A}\alpha})^{-1}$. If a bank chooses not to default, then the quantity of its issues, D, is bounded from above and it gets a finite profit. But if it chooses to issue $D=\infty$, thus to default in the bad state, then its profit is given by (38). It follows that $\Pi=\infty$ because $\widehat{R}/R^{SB}-1>0$ and $\overline{A}\alpha-(1-q)\underline{A}\widehat{R}>0$ for the range of $\widehat{R}$ given in the condition. With $D=\infty$, $r=\frac{q\widehat{R}}{1-(1-q)\underline{A}/(\overline{A}\alpha)\cdot\widehat{R}}-1$ by (37). Intuitively, with $\widehat{R}>R^{SB}$, the profit margin of bank issuance is positive and banks want to issue $D=\infty$, thereby obtaining $\Pi=\infty$, but differently from the case where $\widehat{R}\geq\frac{\overline{A}\alpha}{\underline{A}}$, now with a lower $\widehat{R}$ the issuance of $D=\infty$ cannot be supported by loans only and has to induce default in the bad state. ■

These two lemmas imply result (i) of the proposition. Result (ii) is proved in the following lemma.

Lemma 3C: if $\widehat{R}=R^{SB}$, then the bank obtains $\Pi=0$ and is indifferent to any quantity of issues, D, with r determined by D through equation (36).

Proof. In this case, if banks choose not to default, so that $\delta=1$, then $r=\widehat{R}-1=r^{SB}$. Thus, by the upper branch of (34), the coefficient before D, namely, the profit margin of issuance, equals 0. The same is true if banks choose to default because, then, their profit is given by (38) and the coefficient before D equals 0 too. Therefore, banks obtain 0 profit always and are indifferent to any D. At any chosen D, the value of r is determined by (36), with $\delta(D,r)$ given by (14). ■

Result (iii) is proved in the following lemma.

Lemma 3D: if $\widehat{R}<R^{SB}$, then lending makes banks lose and $D=\Pi=0$.

Proof. In this case, if banks choose not to default, then $r = \widehat{R} - 1 < r^{SB}$. Hence, the coefficient before D in the upper branch of (34) is negative, and lending makes a loss. If banks choose to default, then their profit is given by (38). The coefficient before D, as $\widehat{R} < R^{SB}$, is now negative also, and lending makes a loss too. Therefore, the optimal $D = 0$, whereby $\Pi = 0$. ■

The whole proposition is proved. Q.E.D.

Of Proposition 3:

(i): It has been shown in the main text.

(ii): Prove the "if" part by reduction to absurdity. Suppose that in one equilibrium, no banks default, namely, (13) is honored for all banks. Then, for all banks $\delta = 1$ and, thus, by (36), $1 + r = \widehat{R}|_{\text{result }(i)} = R^{SB}$. By (13), each bank issues,

$$D \leq G/(1 - \frac{\underline{A}}{\overline{A}\alpha}R^{SB})|_{(11)} = \frac{G[q\overline{A}\alpha + (1-q)\underline{A}]}{q(\overline{A}\alpha - \underline{A})}.$$

With $\delta = 1$ and $1 + r = R^{SB}$, by (6) the demand by entrepreneurs is $E = (q\overline{A}\alpha + (1 - q)\underline{A})^{\frac{1}{1-\alpha}} w^{\frac{-\alpha}{1-\alpha}}$. If $G < G^{SB}$, then $D < E$, that is, the supply is below the demand – thus not in equilibrium.

To prove the "only if" part, it suffices to show if $G \geq G^{SB}$, in the symmetric equilibrium, no banks default. Suppose, for the construction of the equilibrium, that it is the case. Then, $\delta = 1$. By the analysis above, $r = R^{SB} - 1$ and entrepreneurs' demand for notes $E = (q\overline{A}\alpha + (1-q)\underline{A})^{\frac{1}{1-\alpha}} w^{\frac{-\alpha}{1-\alpha}}$, to which the notes supply, D, is equal in equilibrium. With this value of (D, r), it is straightforward to check back that if $G \geq G^{SB}$, indeed the no-default condition, (13), is honored and, hence, no banks default in the bad state. Q.E.D.

Of Proposition 4:

Note that as there is no default, $\delta = 1$ and $1 + r = R$.

(i): If $G \geq G^{SB}$, by Proposition 3, there is an equilibrium in which no banks default in the absence of no-default constraint, that is, the constraint is non-binding. Therefore, its presence affects nothing of the real allocation, which is thus the same as in that equilibrium, given by Proposition 3(i), namely, $\widehat{R} = R^{SB}$, the profit margin of bank issuance is 0, and $L = L^{SB}$.

If $G < G^{SB}$, first, note that in the equilibrium, $\widehat{R} > R^{SB}$; otherwise, by Proposition 2, $\widehat{R} = R^{SB}$ (because in no equilibria $\widehat{R} < R^{SB}$, which, by that proposition, discourages banks from lending), at which to meet the demand for banks' issues entails bank default, disallowed here. Second, with $1 + r = R$, no-default constraint (21) becomes

$$D \leq G(1 - \frac{\underline{A}}{\overline{A}\alpha} R)^{-1}. \tag{39}$$

Third, as banks do not default, their profit is given by the upper branch of (34), which, as $r - r^{SB} = (\widehat{R} - 1) - (R^{SB} - 1) = \widehat{R} - R^{SB}$, becomes $\frac{q\overline{A}\alpha + (1-q)\underline{A}}{\overline{A}\alpha}(R - R^{SB})D$.

To maximize the profit, banks' problem is now

$$\max_{D,R} (R - R^{SB})D, \text{ s.t. (39) and } R \leq \widehat{R}.$$

As $\widehat{R} > R^{SB}$, at the maximum, both constraints of this problem are binding. Therefore, the supply of banks' notes is

$$D = \frac{G}{1 - \underline{A}\widehat{R}/(\overline{A}\alpha)}. \tag{40}$$

Intuitively, taking the positive profit margin (which is in proportion to $R - R^{SB} > 0$) as given, banks issue as much as possible, until the no-default constraint, (39), is binding.

The equilibrium real interest rate, $\widehat{R}$, is found by equalizing this supply of banks' notes to the demand, given by (6). With $1 + r = \widehat{R}$ and $\delta = 1$, this equalization gives rise to (22):

$$G = G(\widehat{R}) \equiv (\frac{\overline{A}\alpha}{w^{\alpha}})^{\frac{1}{1-\alpha}} \cdot \frac{1 - \frac{\underline{A}}{\overline{A}\alpha}\widehat{R}}{\widehat{R}^{\frac{1}{1-\alpha}}}.$$

It is straightforward that $G'(\widehat{R}) < 0$, $G(R^{SB}) = G^{SB}$ and $G(\overline{A}\alpha/\underline{A}) = 0$. Therefore, for any given $G < G^{SB}$, equation (22) determines a unique $\widehat{R} \in (R^{SB}, \overline{A}\alpha/\underline{A})$, which, by (40), determines a unique D. Hence, the equilibrium uniquely exists.

(ii): Let $\widehat{R}(G)$ be the inverse function of $G(\widehat{R})$. Then, in the equilibrium $\widehat{R} = \widehat{R}(G)$. As $G(\widehat{R})$ is decreasing, so is $\widehat{R}(G)$. Moreover, $\widehat{R}(G^{SB}) = R^{SB}$ and $\widehat{R}(0) = \overline{A}\alpha/\underline{A}$ because $G(R^{SB}) = G^{SB}$ and $G(\overline{A}\alpha/\underline{A}) = 0$. With $R = \widehat{R}$ for each bank, the number of workers hired in equilibrium, by (7), is $L = (\overline{A}\alpha/w\widehat{R})^{\frac{1}{1-\alpha}}$, which decreases with $\widehat{R}$. Thus, L increases with G. Moreover, $L = L^{SB}$ at $G = G^{SB}$, where $\widehat{R} = R^{SB}$.

Q.E.D.

Of Proposition 5:

(i) The equilibrium is characterized as follows. Entrepreneurs' demand for shells, E_s, solves the same problem as the demand for banks' notes, given by (5), except that r is replaced with r_s and δ with δ_s. Therefore, the real interest rate of borrowing shells is $(1+r_s)/\delta_s$, while the real interest rate of borrowing banks' notes is $1+r$ (since $\delta = 1$ as banks do not default). By (8), only the real interest rate of the borrowing concerns the entrepreneurs. In equilibrium, as made clear later, lending of both means of payment bears a positive profit margin and thus both are lent out. Then, the real interest rate of lending the two are equal (so that both are demanded by entrepreneurs), and, as before, equal to $\widehat{R}$:

$$\frac{1+r_s}{\delta_s} = 1 + r = \widehat{R}. \tag{41}$$

As the profit margin of lending is positive, banks issue notes to the point at which the no-default constraint, (39), becomes binding, that is, $D = G/(1 - \frac{\underline{A}}{\overline{A}\alpha}\widehat{R})$; and also they lend out all the shells borrowed from the central bank, S. The aggregate value of these means of payment supplied, $\delta_s S + D$, when the market clears, equals the wage payment that entrepreneurs demand to hire workers:

$$wL = \delta_s S + \frac{G}{1 - \underline{A}\widehat{R}/(\overline{A}\alpha)}. \tag{42}$$

By (7), the number of workers hired is

$$L = \left(\frac{\overline{A}\alpha}{w}\right)^{\frac{1}{1-\alpha}} \widehat{R}^{\frac{-1}{1-\alpha}}. \tag{43}$$

These four equations (note 41 has two) together with equation (23) (which settles δ_s), as shown below, determine a unique profile of $(\delta_s, r_s, r, \widehat{R}, L)$ in equilibrium – thus, the equilibrium exists uniquely. Passing on to show that, we derive equations (24) and (25). By (41), $1 + r_s = \widehat{R}\delta_s$. Substituting it into (23) and rearranging, we have:

$$\delta_s = \frac{q}{1 - (1-q)\underline{A}/(\overline{A}\alpha) \cdot \widehat{R}}. \tag{44}$$

Substitute it and (43) into (42)), rearrange, let $\theta \equiv \underline{A}/(\overline{A}\alpha) \cdot \widehat{R}$, and we come to (24):

$$S = (\frac{\underline{A}}{w^{\alpha}})^{\frac{1}{1-\alpha}} \frac{1-(1-q)\theta}{q\theta^{\frac{1}{1-\alpha}}} - \frac{1-(1-q)\theta}{q(1-\theta)} G \equiv F(\theta). \tag{45}$$

Equations (41) and (44) together imply $1 + r_s = \frac{q\widehat{R}}{1-(1-q)\underline{A}/(\overline{A}\alpha)\cdot\widehat{R}}$, from which (25) follows.

Now, come to show that for $S \leq \underline{S}(G)$, equation (45) determines a unique $\widehat{R}$ (which determines a unique r_s, r, L, and δ_s); and that this $\widehat{R}$ decreases from $\widehat{R}(G)$ to R^{SB} when S runs from 0 to $\underline{S}(G)$, where $\widehat{R}(G)$ is the inverse function of $G(\widehat{R})$ given by (22) in Proposition 4, namely, the equilibrium real interest rate without central-bank intervention. Both assertions follow from the following three observations. First, $F'(\theta) < 0$, therefore, θ, and thus $\widehat{R}$, decreases with S. Second, equation $F(\theta) = 0$ is equivalent to (22), thus leading to $\widehat{R} = \widehat{R}(G)$. Therefore, at $S = 0$, $\widehat{R} = \widehat{R}(G)$. And third, at $\widehat{R} = R^{SB}$, $\theta = \frac{\underline{A}}{\overline{A}\alpha} R^{SB}$ and $F(\frac{\underline{A}}{\overline{A}\alpha} R^{SB}) = \underline{S}(G)$. At $S = \underline{S}(G)$, therefore $\widehat{R} = R^{SB}$.

Lastly, $\underline{A}(1 + r_s)/(\overline{A}\alpha) < 1 \Leftrightarrow 1 + r_s < \overline{A}\alpha/\underline{A}|_{(25)} \Leftrightarrow q\widehat{R}/[1 - (1-q)\underline{A}/(\overline{A}\alpha) \cdot \widehat{R}] < \overline{A}\alpha/\underline{A} \Leftrightarrow \widehat{R} < \overline{A}\alpha/\underline{A}|_{\widehat{R}<\widehat{R}(G)} \Leftarrow \widehat{R}(G) < \overline{A}\alpha/\underline{A}$, which is affirmed by Proposition 4(ii).

(ii): It was shown above that $\widehat{R}$ decreases with S and equals R^{SB} at $S = \underline{S}(G)$. By (25), r_s increases with $\widehat{R}$ and $r_s = 0$ at $\widehat{R} = R^{SB}$. Therefore, r_s decreases with S and equals 0 at $S = \underline{S}(G)$.

(iii): In the unique equilibrium, each bank serves N entrepreneurs and obtains from each of them $\widehat{\pi}$, given by (18), thus $N\widehat{\pi}$ overall. Note that $\widehat{\pi}$ increases with $\widehat{R}$ for $\widehat{R} \in [R^{SB}, \frac{1}{\alpha}R^{SB}]$ and decreases with it for $\widehat{R} > \frac{1}{\alpha}R^{SB}$. Then, the result follows from the fact that $\widehat{R} \leq \widehat{R}(G) \leq \frac{1}{\alpha}R^{SB}$ for any $S \geq 0$ if $G \geq G(\frac{1}{\alpha}R^{SB})$ and that $\widehat{R} = \frac{1}{\alpha}R^{SB}$ at $S = F(\frac{\underline{A}}{\overline{A}\alpha} \cdot \frac{1}{\alpha}R^{SB})$ if $G < G(\frac{1}{\alpha}R^{SB})$.

(iv): As $G < G^{FB}$, $\widehat{R}(G) > R^{FB} > R^{SB}$. Note that $\widehat{R} = \widehat{R}(G)$ at $S = 0$ and $\widehat{R} = R^{SB}$ at $S = \underline{S}(G)$. Therefore, there is a unique S between 0 and $\underline{S}(G)$ at which $\widehat{R} = R^{FB}$ and this S equals $F(\frac{\underline{A}}{\overline{A}\alpha} \cdot R^{FB})$ by (24).

Q.E.D.

Of Proposition 6:

It is proved in the main text.

Of Proposition 7:

(i): As was said, the real interest rate in equilibrium is determined by three conditions: (28); (29); and if (29) is not binding, then (28) is. Note that (29) is equivalent to

$$1 + r_p \leq R/R^{SB} \tag{46}$$

and (28) is equivalent to

$$1 + r_p \leq \Phi(R), \tag{47}$$

where $\Phi(R)$ is defined in (30):

$$\Phi(R) \equiv \frac{Gw^{\frac{\alpha}{1-\alpha}}}{(\overline{A}\alpha)^{\frac{1}{1-\alpha}}} R^{\frac{1}{1-\alpha}} + \frac{\underline{A}}{\overline{A}\alpha} R.$$

Then in equilibrium, (47) and (46) hold, and one of them must be binding.

The binding constraint is the one that is tighter, namely that with a smaller value on the right hand side. Let $\chi(R) \equiv \Phi(R) - R/R^{SB}$ for $R \geq 0$. Then $\chi(R) = 0$ has two roots: 0 and

$$R^* = \frac{\overline{A}\alpha}{w} [\frac{q(\overline{A}\alpha - \underline{A})}{G}]^{\frac{1-\alpha}{\alpha}}.$$

$R^* \geq R^{SB}$ if and only if $G \leq G^{SB}$. With $\chi'(0) < 0$, we have $\chi < 0$ for $R \in (0, R^*)$ and $\chi > 0$ for $R > R^*$. Moreover, let $r_p^* \equiv R^*/R^{SB} - 1 = [q(\overline{A}\alpha - \underline{A})/G]^{\frac{\alpha}{1-\alpha}}(q\overline{A}\alpha + (1 -$

$q)\underline{A})/w - 1$, which, by the definition of R^*, also equals $\Phi(R^*) - 1$. Finally, as $r_p \geq 0$, $R \geq R^{SB}$ by (46).

Consider first the case where $G < G^{SB}$. In this case, $R^{SB} < R^*$. By the definition of R^*, for $R \in [R^{SB}, R^*)$, $\chi < 0$, and thus $\Phi(R) < R/R^{SB}$, therefore, (47) is binding. For $R > R^*$, $\chi > 0$, and thus $\Phi(R) > R/R^{SB}$, therefore, (46) is binding. In both cases, R increases with r_p, and at $R = R^*$, $r_p = r_p^*$. Therefore, $R < R^*$ if $r_p < r^*$ and $R > R^*$ if $r_p > r^*$. It follows that if $r_p < r^*$, as $R < R^*$, (47) is binding, giving rise to $1 + r_p = \Phi(R)$ and if $r_p > r^*$, (46) is binding, giving rise to $R = R^{SB}(1 + r_p)$.

Consider then the case where $G \geq G^{SB}$, and thus $R^{SB} \geq R^*$. For all $R \geq R^{SB}$ (namely, all the possible equilibrium values of R), $R \geq R^*$. Then, always $\chi > 0$ and thus $\Phi(R) > R/R^{SB}$. Therefore, (46) is always binding, giving rise to $R = R^{SB}(1 + r_p)$.

(ii): There exists a unique policy rate under which $R = R^{FB}$ and the first-best allocation is attained: by Proposition 4, at $r_p = 0$ (namely, without intervention by the CB), $R < R^{FB}$ because now $G > G^{FB}$; by result (i), R increases with r_p to infinite; therefore, there exists a unique r_p under which $R = R^{FB}$. To find this r_p, it suffices to find the inverse function of $R(r_p)$ given by result (i). This inverse function is that for $G < G^{SB}$, $r_p = \Phi(R) - 1$ if $R \leq \Phi^{-1}(1 + r_p^*) = R^*$ and $r_p = R/R^{SB} - 1$ otherwise, while for $G \geq G^{SB}$, $r_p = R/R^{SB} - 1$ always. $R^* > R^{FB}$ if and only if $G < G_s$. Therefore, R^{FB} falls in the domain of function $\Phi(R) - 1$ if $G < G_s$, otherwise it is in the domain of function $R/R^{SB} - 1$. Hence, the optimal policy rate, r_p^{FB}, equals $\Phi(R^{FB}) - 1$ if $G < G_s$ and $R^{FB}/R^{SB} - 1$ otherwise. It is straightforward to check $G^{FB} < G_s < G^{SB}$.

Q.E.D.

Of Proposition 8:

We first calculate the debt to equity ratio. The value of a bank's equity is the difference between the asset value and liability, $G + Y - D$. By the discussion preceding (16), the value of the bank's loans at $t = 1$ is $D(1 + r)$ in the good state and is

$\underline{A}(1+r)/(\overline{A}\alpha) \times D$ in the bad state. Therefore, at $t = 0$,

$$Y = q \times D(1+r) + (1-q) \times \frac{\underline{A}(1+r)}{\overline{A}\alpha} D.$$

As banks do not default, $1 + r = R$. At the optimum, banks choose $R = \widehat{R}$. Therefore,

$$Y = [q\widehat{R} + (1-q)\frac{\underline{A}\widehat{R}}{\overline{A}\alpha}] \times D.$$

Thus the leverage of the bank is

$$l = \frac{D}{G + [q\widehat{R} + (1-q)\frac{\underline{A}\widehat{R}}{\overline{A}\alpha} - 1] \times D}.$$

By this formula, at the first best allocation where each bank issues $D = D^{FB}$ and thereby $\widehat{R} = R^{FB}|_{(10)} = \frac{\overline{A}}{A_e}$, the leverage of banks is $l^{FB} = \frac{D^{FB}}{G + \frac{(1-q)\underline{A}(1-\alpha)}{A_e \alpha} D^{FB}}$. Moreover, notice that l increases with D: $l = [\frac{G}{D} + q\widehat{R} + (1-q)\frac{\underline{A}\widehat{R}}{\overline{A}\alpha} - 1]^{-1}$ and $\widehat{R}$ (the interest rate of lending) decreases with D (the scale of lending). It follows that $l \leq l^{FB}$ if and only if $D \leq D^{FB}$, which is equivalent to $\frac{D}{G} \leq \frac{D^{FB}}{G}$. This constraint, we have shown in the main text, is binding and attains the first-best allocation.

Q.E.D.

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