A Model of Leverage Based on Risk Sharing

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Abstract

This paper offers a new approach, based on risk sharing, to endogenize the leverage of financial intermediaries. It endogenizes debt as the optimal contract for external financing, thereby capturing two features of leverage: debt serves to boost the return on equity; and equity provides "safety net" for debt. The paper derives a novel prediction that when the asset-side risk rises, the leverage ratio is reduced, but the profit margin of leveraging is actually widened.

Key Words: Risk Sharing; Leverage; Financial Intermediaries

JEL: D86, G20, G30

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1 Introduction

Leverage is ubiquitously used in the financial sector for boosting investment performance. For example, a study by British Private Equity & Venture Capital Association and Ernest & Young finds that half of the average return of the 14 biggest private equity deals realized in 2005-7 "came from the use of extra debt...."[1] This paper offers a new approach to study how financial intermediaries, such as banks and private equity funds, decide on leverage when seeking to boost their investment performance.

The approach here is based on risk sharing and can be illustrated as follows. Suppose a risk neutral bank and many extremely risk averse households each have \$1 to invest. There are two investment channels. One is risk free with gross return rate 1. The other is risky with a gross return rate of 0.9 or 1.3, each with probability one half, so the expected rate is 1.1. The bank, being risk neutral, puts its dollar in the risky asset earning an expected revenue of \$1.1, which gives a modest 10% net return rate. However, the bank can do better by taking in households' funds. Suppose it takes in one household's dollar and invests the entire \$2 (the taken-in one plus its own one) in the risky asset. The investment returns $\$2 \times 0.9 = \$1.8$ in the bad state and $\$2 \times 1.3 = \$2.6$ in the good state. The household is satisfied with getting \$1 back in both states. The bank thus earns, with this minor leverage, an expected revenue of $0.5 \times \$(1.8 - 1) + 0.5 \times \$(2.6 - 1) = \$1.2$, \$0.1 more than it earns without leverage. This extra \$0.1 is the difference between \$1.1, the return on one dollar investment on the asset side, and \$1, the repayment on the liability side to the household. Note that this repayment satisfies the household because it is risk free, which in turn is because his claim is senior to the bank's. That is, the contract to the household is debt and the bank's dollar forms the equity acting as the cushion to absorb loss to the household. The optimal amount of borrowing is \$9. At this level, the bad state revenue, $\$(1 + 9) \times 0.9 = \$9$, exactly suffices to service the risk-free debt. With this leverage, the bank earns \$2 for its dollar and achieves a shining net return rate of 100%. Thus the approach captures how leverage can be used to boost the performance of equity (i.e. the bank's dollar). Moreover, it explains why debt is cheap, namely, why the debt holders are willing to accept a return rate lower than prevails on the asset side, enabling debt to be used to raise the rate of return on equity.

[1] See "Buy-out Profits Tied to Debt", *Financial Times*, January 15, 2009.

This approach could be regarded dual to that used by Gale (2003, 2004) and Gale and Ozgur (2005): In the present paper, the risk averse agents get the reserved utility (which is endogenized) and the profit of the risk neutral agents is maximized, while in their papers, the risk neutral agents get their reservation profit and the utility of those risk averse is maximized. With this innovation, the approach here captures how leverage is used to boost the performance of equity investment.

There are other approaches to leverage decisions. The leverage ratio is determined by a Value-at-Risk rule in, e.g., Brunnermeier and Pedersen (2008) and Liu and Mello (2008), or by non-default requirement in, e.g. Geanakoplos and Polemarchakis (1986), Geanakoplos (1997), Fostel and Geanakoplos (2008), or driven by risk-shifting concerns in, e.g., Adrian and Shin (2008). Compared to these approaches, the approach here endogenizes debt as the optimal contract for external financing, and explains why it is cheap.

Moreover, in contrast with all the above literature, the paper derives a novel prediction, that when the asset-side risk rises, the profit margin from leveraging is actually widened, though the leverage ratio is reduced.

The rest of the paper is organized as follows. Section 2 sets up the model and then analyzes it. Section 3 concludes. All the proofs are relegated to Appendix.

2 The Model

There are a continuum of 1 unit banks and that of N units of households. Banks are risk neutral and protected by limited liability. For $i \in [0,1]$, bank i has $\$K_i$ of funds. Let $K \equiv \int_0^1 K_i$ denote the aggregate supply of banks' funds. Households are risk averse, their preferences represented by a strictly concave function $U(\cdot)$, with $U'(\cdot) > 0$ and $U''(\cdot) < 0$, and each has a small amount of funds to invest, normalized to \$1.

Funds are either put into 1-to-1 storage or invested in a risky asset, of which the gross rate of return is $\overline{R}$ with probability q and $\underline{R}$ with probability $1-q$, where $\overline{R} > \underline{R}$. Assume:

$$R_e \equiv q\overline{R} + (1-q)\underline{R} > 1 \tag{1}$$

$$qU(\overline{R}) + (1-q)U(\underline{R}) < U(1). \tag{2}$$

Furthermore, to simplify the exposition, I assume that households cannot split their dollars

between two channels of investment.

By (1), banks invest all their funds in the risky asset and earn a return rate of R_e. However, as was illustrated in the Introduction, banks can boost the performance of investment by borrowing from households. Let L_i denote the amount of bank i's borrowing and $L \equiv \int_0^1 L_i$ denote the aggregate demand of households' funds by banks The market where banks compete for households' funds, by the non-split assumption, is cleared by a *certainty equivalent return rate,* denoted by r, such that a household gets payoff $U(r)$ if investing in a bank rather than storing its dollar. The competitive equilibrium is thus defined as follows.

Definition 1 *A profile of* $(r; \{L_i\}_{i\in[0,1]})$ *forms a competitive equilibrium if:*

(i) given r*, bank* i *takes in* $\$L_i$ *from the households;*

(ii) if $r > 1$*, then* $L = N$*; and if* $0 < L < N$*, then* $r = 1$*.*

Conditions (i) is self evident. Condition (ii) presents how the capital market is cleared: if $r > 1$, then all households prefer investing in banks to storage and thus the market is cleared at $L = N$; if only part of households invest in banks, namely, $0 < L < N$, then households must be indifferent between such investment and storage, that is, $r = 1$.

Below I first examine how a typical bank decides on its leverage (i.e. condition i), then moving on to market clearing (i.e. condition ii).

2.1 The Leverage of the Representative Bank

Suppose bank i, which has $\$K_i$ of its own funds, takes in L_i households' funds by issuing a security that promises to repay each of them $\overline{h}$ in the good state (when $\overline{R}$ is realized) and $\underline{h}$ in the bad state (when $\underline{R}$ is realized). Let $h_e \equiv q\overline{h} + (1-q)\underline{h}$. The bank invests all $K_i + L_i$ of funds in the risky asset, which returns $(K_i + L_i)\overline{R}$ in the good state and $(K_i + L_i)\underline{R}$ in the bad one.[2] The total liability repayment is $L_i\overline{h}$ and $L_i\underline{h}$ respectively. Hence, the bank's expected profit is $\Pi = q((K_i + L_i)\overline{R} - L_i\overline{h}) + (1-q)((K_i + L_i)\underline{R} - L_i\underline{h})$.

The problem of the bank is thus to find $\{L_i, \overline{h}, \underline{h}\}$ that maximizes Π subject to the following individual rationality (IR) and limited liability constraints.

[2] Intuitively, the bank will not put any funds in storage because the profit margin of doing so is $1 - h_e < 0$.

Firstly, security $(\overline{h}, \underline{h})$ must give the investing households payoff $U(r)$ which they could get by investing in other banks (or from storage if $r = 1$), namely

$$qU(\overline{h}) + (1-q)U(\underline{h}) = U(r). \tag{3}$$

Secondly, in both states, the investment revenue suffices to cover the liability outlay:

$$L_i\overline{h} \leq (K_i + L_i)\overline{R} \tag{4}$$

$$L_i\underline{h} \leq (K_i + L_i)\underline{R}. \tag{5}$$

The solution of the problem is characterized as follows.

Proposition 1 *(i) If $r > R_e$, then $L_i = 0$.*

(ii) If $r = R_e$, then the bank is indifferent with any $L_i \in [0, \underline{R}/(R_e - \underline{R}) \cdot K_i]$. If $L_i > 0$, then $\overline{h} = \underline{h} = R_e$.

(iii) If $r < R_e$, then $L_i = lK_i$, where $\{l, \overline{h}, \underline{h}\}$ are determined by the simultaneous equations of (3) and

$$\underline{h} = \frac{(1+l)\underline{R}}{l} \tag{6}$$

$$\overline{R} - \overline{h} - \frac{(1-q)U'(\underline{h})}{qU'(\overline{h})}\frac{\underline{R}}{l} = 0. \tag{7}$$

Moreover, the security to the households is risky: $\underline{h} < r < \overline{h}$.

Proof. See Appendix. ■

Results (i) and (ii) are intuitive. The profit margin to the bank of taking in households' funds is $R_e - h_e$. The households, being risk averse, demand a risk premium of $h_e - r \geq 0$. Therefore, if $r > R_e$, banks cannot profit from households' funds. Hence result (i). And if $r = R_e$, the profit margin of leverage is at most 0, which occurs only when $h_e = r$, namely with the risk free security: $\overline{h} = \underline{h} = R_e$, which, by (5), implies $L_i \leq \underline{R}/(R_e - \underline{R}) \cdot K_i$. This is result (ii).

If $r < R_e$, so long as the bank can offer the risk free security ($\overline{h} = \underline{h} = r$), the profit margin is $R_e - r > 0$. With this security, the bank wants to get as much of households' funds as possible, until (5) is binding, which gives rise to (6) (with $l \equiv L_i/K_i$). Moreover, at the optimum, the security is risky. Start with the risk free contract and now introduce a little risk to it: $\underline{h} = r - \epsilon$ and $\overline{h} = r + (1-q)/q\epsilon + o(\epsilon)$, which allows for L_i to increase by δL_i, in an order of ϵ, but still

keeps the IR, (3), satisfied. This variation benefits the bank: the extra cost is of the second order ($qo(\epsilon)$), whereas the gain is $\delta L \cdot (R_e - r - qo(\epsilon))$, of the first order.[3] Lastly, the optimal leverage ratio, L_i/K_i, is independent of K_i, because for the bank's contracting problem both the objective Π and the constraints depend on L_i only through the ratio of L_i/K_i.

As (5) is binding, the bank uses all the revenue in the bad state to pay the households, namely, *the optimal contract to households is debt.* And the debt is risky.

Below I discuss properties of the optimal leverage ratio given in result (iii), which is a function of $\overline{R}$, $\underline{R}$ and r and thus denoted by $l(\overline{R}, \underline{R}, r)$. Intuitively, the ratio is driven by a trade-off between the leverage scale (L_i) and the profit margin, $R_e - h_e$. On the one hand, given the profit margin, the more the bank borrows, the more it profits. On the other hand, the more a bank borrows, the less the equity capital available to cushion each borrowed dollar and thus the bigger the risk posed to the debt holders, who thus demand a higher risk premium (i.e. $h_e - r$), leading to a smaller profit margin.

The asset-side risk can be measured by the standard variation of the return rate, $\sqrt{q(1-q)}(\overline{R} - \underline{R}) \equiv \sigma$.[4]

Proposition 2 $\frac{\partial l}{\partial r} < 0$, $\frac{\partial l}{\partial \overline{R}} > 0$ *and* $\frac{\partial l}{\partial \underline{R}} > 0$. *Moreover,* $\frac{\partial l}{\partial \sigma} < 0$.

Proof. See Appendix. ■

The intuition behind the proposition is that a variation that enlarges the profit margin ($R_e - h_e$) at any leverage scale will engender scope to increase the leverage ratio (l). As such, an increment in r certainly reduces the profit margin, thus diminishing l, whereas l is decreased by a rise in any return rate ($\overline{R}$ or $\underline{R}$) which increases R_e and thus the profit margin. And $\frac{\partial l}{\partial \sigma} < 0$ is because if σ increases, with R_e fixed, then the profit margin falls since h_e is increased, which in turn is because at any given L_i, a higher risk on the asset side transforms into a higher risk on the liability side, whence the debt holders demand a greater risk premium, $h_e - r$.

In the paper, against \$1 of investment in the risky asset, the bank borrows $L/(K+L) = l/(1+l)$ from the households and contributes $1/(1+l)$ of its own. Hence the "haircut" rate

[3] I thank the referee for the intuition.

[4] Namely, a function of $(\overline{R}, \underline{R})$ becomes a function of (δ, R_e) through the linear transformation: $\overline{R} = R_e + \sqrt{\frac{1-q}{q}}\delta$ and $\underline{R} = R_e - \sqrt{\frac{q}{1-q}}\delta$.

is $1/(1+l)$. Then from $\frac{\partial l}{\partial \sigma} < 0$ it follows that the "haircut" rate increases with σ. That gives an explanation for the phenomenon that a bigger haircut is imposed when the asset risk or uncertainty is higher, a phenomenon widely noticed by practitioners and scholars such as Gorton and Metric (2010).

The expected return rate of the banker's capital, π, defined as Π/K, equal $R_e + l \cdot (R_e - h_e)$, as with each dollar of its own, the bank gains $\$R_e$ directly as the investment revenue and borrows from households l dollars each of which earns profit margin $R_e - h_e$.

Proposition 3 *Given R_e, $\frac{d\pi}{d\sigma} < 0$, that is, the expected rate of return on the bank equity decreases with the risk on the asset side.*

Proof. See Appendix. ■

The proposition assets that *although the bank is risk neutral on its own, it becomes risk averse for the sake of its household debt holders.*

The proposition is driven by $\frac{\partial l}{\partial \sigma} < 0$, that is, a high asset risk reduces the bank's capability of using leverage to boost its return on equity. Actually, as $\pi = R_e + l \cdot (R_e - h_e)$, it might be conjectured that the other channel for π to decrease with σ is that the margin, $R_e - h_e$, also narrows down with a higher asset risk. That, however, is not the case:

Proposition 4 $\frac{\partial (R_e - h_e)}{\partial \sigma} > 0$, *that is, the profit margin of leveraging increases with the risk on the asset side.*

Proof. See Appendix. ■

The reason for the proposition is that with the asset-side risk increased, the leverage scale is reduced so much that the risk passed on to the debt holders is actually reduced, thus so is the demanded risk premium $(h_e - r)$. The proposition is peculiar to the paper's approach, in which the cost of debt to the bank, h_e, depends on the leverage ratio and thus on the asset risk. It would not be implied by the other approaches in the literature (discussed in the Introduction), where the cost of debt is exogenous and independent of the asset risk, thus so is the profit margin, $R_e - h_e$.

This proposition, together with $\frac{\partial l}{\partial \sigma} < 0$, predicts that *when the asset-side risk rises, the leverage ratio is reduced, but the profit margin from leveraging is actually widened.*

2.2 The Equilibrium: How the Surplus is Shared

By Proposition 1, the aggregate demand of households' funds by banks is

$$L(r) = \left\{ \begin{array}{c} 0, \text{ for } r > R_e \\ [0, \underline{R}/(R_e - \underline{R}) \cdot K], \text{ at } r = R_e \\ l(\overline{R}, \underline{R}, r) \cdot K, \text{ for } R_e > r \geq 1 \end{array} \right\},$$

while the aggregate supply is N. The following proposition shows that the division of surplus from leverage between banks and households is determined by the relative aggregate strength of the two sides, measured by N/K.

Proposition 5 *(i) If $N \leq \underline{R}/(R_e - \underline{R}) \cdot K$, then $r = R_e$ and all the surplus goes to households.*

(ii)If $l(\overline{R}, \underline{R}, 1) \cdot K < N < \underline{R}/(R_e - \underline{R}) \cdot K$, then $1 < r < R_e$ and the surplus is shared between banks and households. Moreover, r strictly decreases with N/K.

(iii) If $N > l(\overline{R}, \underline{R}, 1) \cdot K$, then $r = 1$ and all the surplus goes to banks.

All is straightforward, except the strictly decreasing of r with N/K in result (ii), which follows from $\partial l/\partial r < 0$ given in proposition 2.

3 Conclusion

The paper provides a new approach to the leverage decisions of financial intermediaries. This approach is based on risk sharing between agents differing in risk preference. It endogenizes debt as the optimal contract for external financing and captures two features: (i) equity serves as a "safety net" for debt in rainy days; and (ii) debt boosts the return on equity on average. These two features are intertwined: it is because of being protected by equity that debt is cheap enough to be used to boost equity. Moreover, the paper derives a novel prediction that when the asset-side risk rises, the profit margin of leverage is actually widened, though the leverage ratio is reduced. Finally, the paper shows that the division of surplus from leverage between financial intermediaries and their lenders is determined by the relative aggregate strength of the two sides.

Acknowledgement: I am indebted to John Moore for the long discussions with him and to Roy Bailey for his help with exposition. I also thank Sanjay Banerji, Sudipto Dasgupta, Leonardo Felli, Gorden

Kemp, Xuewen Liu, Motty Perry, Wolfgang Pesendorfer, Francesco Squintani, and Asher Wolinsky, as well as seminar participants in various places, for their helpful comments.

Appendix: The Proofs

Proof of Proposition 1:

For (i) and (ii) see the main text. For (iii), note that the bank's problem is equivalent to maximize: $\pi \equiv \Pi/K_i = R_e + l(R_e - h_e)$, with $l \equiv L_i/K_i$, subject to (3), and (a) $l\overline{h} \leq (1+l)\overline{R}$ and (b) $l\underline{h} \leq (1+l)\underline{R}$ (equivalent to 4 and 5). The Lagrangian is thus $\mathcal{L} = R_e + l(R_e - h_e) + \lambda(qU(\overline{h}) + (1-q)U(\underline{h}) - U(r)) + \overline{\mu}((1+l)\overline{R} - l\overline{h}) + \underline{\mu}((1+l)\underline{R} - l\underline{h})$, with $\lambda, \overline{\mu}, \underline{\mu}$ all non-negative. Then, from $(1-q)\frac{\partial\pi}{\partial\overline{h}} - q\frac{\partial\pi}{\partial\underline{h}} = 0$ it follows $ql\underline{\mu} = \lambda q(1-q)(U'(\underline{h}) - U'(\overline{h})) + (1-q)l\overline{\mu} \geq \lambda q(1-q)(U'(\underline{h}) - U'(\overline{h})) > 0$. Therefore, constraint (a) is binding, which gives rise to (6). It and (3) implicitly determines $(\overline{h}, \underline{h})$ as functions of l. Taking this into account, (7) follows from $\frac{d\pi}{dl} = 0$. Q.E.D.

Proof of Proposition 2:

The leverage ratio, l, is determined by $F(\overline{R}, \underline{R}, r, l) \equiv \overline{R} - \overline{h} - \frac{(1-q)U'(\underline{h})}{qU'(\overline{h})}\frac{\underline{R}}{l} = 0$ (i.e. equation 7) where $\underline{h}$ and $\overline{h}$ are functions of $\underline{R}$, r and l implicitly defined by (6) and (3). Let $S(l, \underline{R}) \equiv \frac{(1-q)U'(\underline{h})}{qU'(\overline{h})}$. Then, $\frac{\partial\underline{h}}{\partial r} = 0, \frac{\partial\overline{h}}{\partial r} > 0$, $\frac{d\overline{h}}{d\underline{h}} = -S$, $\frac{\partial\underline{h}}{\partial\underline{R}} = \frac{1+l}{l}$, $\frac{\partial\underline{h}}{\partial l} = -\frac{\underline{R}}{l^2}$, $\frac{\partial\overline{h}}{\partial\underline{R}} = \frac{d\overline{h}}{d\underline{h}} \cdot \frac{\partial\underline{h}}{\partial\underline{R}} = -S\frac{1+l}{l}$, and $\frac{\partial\overline{h}}{\partial l} = \frac{d\overline{h}}{d\underline{h}} \cdot \frac{\partial\underline{h}}{\partial l} = \frac{S\underline{R}}{l^2}$. Moreover $\frac{\partial S}{\partial l} > 0$ and $\frac{\partial S}{\partial\underline{R}} < 0$: With l increasing or $\underline{R}$ decreasing, $\underline{h}$ decreases by (6), then $\overline{h}$ increases by (3), then $U'(\underline{h})$ increases and $U'(\overline{h})$ decreases since $U'(\cdot)$ is decreasing, and hence S increases. Finally, since $\overline{R} = R_e + \sqrt{\frac{1-q}{q}}\sigma$ and $\underline{R} = R_e - \sqrt{\frac{q}{1-q}}\sigma$, we have $\frac{\partial\overline{R}}{\partial\sigma} = \sqrt{\frac{1-q}{q}}$, and $\frac{\partial\underline{R}}{\partial\sigma} = -\sqrt{\frac{q}{1-q}}$. Then, for $F(\overline{R}, \underline{R}, l)$, we have:

$\frac{\partial F}{\partial l} = -\frac{\partial\overline{h}}{\partial l} - \frac{\partial S}{\partial l} \cdot \frac{\underline{R}}{l} + \frac{S\underline{R}}{l^2} = -\frac{S\underline{R}}{l^2} - \frac{\partial S}{\partial l}\frac{\underline{R}}{l} + \frac{S\underline{R}}{l^2} = -\frac{\partial S}{\partial l}\frac{\underline{R}}{l} < 0$, as $\frac{\partial S}{\partial l} > 0$.

$\frac{\partial F}{\partial r} = (-1 + \frac{(1-q)U'(\underline{h})}{qU'(\overline{h})^2}\frac{\underline{R}}{l} \cdot U''(\overline{h})) \cdot \frac{\partial\overline{h}}{\partial r} < 0$.

$\frac{\partial F}{\partial\overline{R}} = 1 > 0$.

$\frac{\partial F}{\partial\underline{R}} = -\frac{\partial\overline{h}}{\partial\underline{R}} - \frac{\partial S}{\partial\underline{R}} \cdot \frac{\underline{R}}{l} - \frac{S}{l} = S\frac{1+l}{l} - \frac{\partial S}{\partial\underline{R}}\frac{\underline{R}}{l} - \frac{S}{l} = S - \frac{\partial S}{\partial\underline{R}}\frac{\underline{R}}{l} > 0$, as $\frac{\partial S}{\partial\underline{R}} < 0$.

$\frac{\partial F}{\partial\sigma} = \frac{\partial F}{\partial\overline{R}} \cdot \frac{\partial\overline{R}}{\partial\sigma} + \frac{\partial F}{\partial\underline{R}} \cdot \frac{\partial\underline{R}}{\partial\sigma} = \sqrt{\frac{1-q}{q}} + (S - \frac{\partial S}{\partial\underline{R}}\frac{\underline{R}}{l}) \cdot (-\sqrt{\frac{q}{1-q}}) = \sqrt{\frac{q}{1-q}}(\frac{1-q}{q} - S) + \sqrt{\frac{q}{1-q}} \cdot \frac{\partial S}{\partial\underline{R}}\frac{\underline{R}}{l} < \sqrt{\frac{q}{1-q}} \cdot \frac{\partial S}{\partial\underline{R}}\frac{\underline{R}}{l} < 0$, where to get the first inequality, we apply $S = \frac{(1-q)U'(\underline{h})}{qU'(\overline{h})} > \frac{1-q}{q}$.

Therefore, $\frac{\partial l}{\partial r} = -\frac{\partial F}{\partial r}/\frac{\partial F}{\partial l} < 0$, $\frac{\partial l}{\partial\overline{R}} = -\frac{\partial F}{\partial\overline{R}}/\frac{\partial F}{\partial l} > 0$, $\frac{\partial l}{\partial\underline{R}} = -\frac{\partial F}{\partial\underline{R}}/\frac{\partial F}{\partial l} > 0$, and $\frac{\partial l}{\partial\sigma} = -\frac{\partial F}{\partial\sigma}/\frac{\partial F}{\partial l} < 0$. Q.E.D.

Proof of Proposition 3:

Note that $\pi = \max_l q[(1+l)\overline{R} - l\overline{h}(l, \underline{R})]$, where $\overline{h}$ is a function of $\underline{h}$ through (3), which in turn is a function of l and $\underline{R}$ through (6). In the above proof of Proposition 2, we found $\frac{\partial \overline{h}}{\partial \underline{R}} = -S\frac{1+l}{l}$, with $S \equiv \frac{(1-q)U'(\underline{h})}{qU'(\overline{h})} > \frac{1-q}{q}$, $\frac{\partial \overline{R}}{\partial \sigma} = \frac{1}{q}$, and $\frac{\partial \underline{R}}{\partial \sigma} = \frac{-1}{1-q}$. By the Envelop Theorem, given R_e, $\frac{d\pi}{d\sigma} = q(1+l)\frac{\partial \overline{R}}{\partial \sigma} - ql \cdot \frac{\partial \overline{h}}{\partial \underline{R}} \cdot \frac{\partial \underline{R}}{\partial \sigma} = q(1+l)\sqrt{\frac{1-q}{q}} - ql \cdot (-S\frac{1+l}{l}) \cdot (-\sqrt{\frac{q}{1-q}}) = (1+l)\sqrt{q(1-q)}(\frac{1-q}{q} - S) < 0$. Q.E.D.

Proof of Proposition 4:

$\pi \equiv \Pi/K = q[(1+l)\overline{R} - l\overline{h}] = q\overline{h} + (1+l)q(\overline{R} - \overline{h})$. Substituting $\overline{R} - \overline{h}$ from (7) and noting $\frac{(1+l)\underline{R}}{l} = \underline{h}$ give:

$$\pi = q\overline{h} + \frac{U'(\underline{h})}{U'(\overline{h})}(1-q)\underline{h} \equiv \pi(\underline{h}), \tag{8}$$

where $\overline{h}$ is a function of $\underline{h}$ implicitly defined by the IR constraint, (3). Then:

Claim: $\pi'(\underline{h}) < 0$.

Proof: Note by (3) $\frac{d\overline{h}}{d\underline{h}} = -\frac{(1-q)U'(\underline{h})}{qU'(\overline{h})} < 0$. Then $\frac{d\pi}{d\underline{h}} = (1-q)\underline{h} \cdot d(\frac{U'(\underline{h})}{U'(\overline{h})})/d\underline{h}$, and the derivative term is negative: with $\underline{h}$ increased, $\overline{h}$ decreases; then $U'(\underline{h})$ decreases and $U'(\overline{h})$ increases; and hence $\frac{U'(\underline{h})}{U'(\overline{h})}$ decreases. Q.E.D.

As $\pi'(\underline{h}) < 0$, and by Proposition 3 $\frac{d\pi}{d\sigma} < 0$, it follows that that $\frac{d\underline{h}}{d\sigma} > 0$. Then $\frac{d\overline{h}}{d\sigma} < 0$, as $\overline{h}$ is inversely related to $\underline{h}$. Thus, with σ higher, the variance of the return rates to the households is smaller, which, by (3), implies a lower expected return rate, h_e. Q.E.D.

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