Reputation and Scale

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Abstract

The paper offers an explanation for why large scales are necessary to establish reputation. In the model, each producer yields one widget and chooses the quality investment each period. Low quality widgets always fail and high quality succeed with some probability. Due to noisiness of single performance, a producer cannot establish reputation in the market. This noisiness problem is resolved by the Law of Large Numbers, as the performance of many widgets is informative of the average quality. When the scale is large, however, defaulting to invest a small number of widgets seems undetectable, which, if true, will break down the equilibrium expectation of the average quality. This incentive problem is resolved by Central Limit Theorem, which ensures increasing marginal return of making the investment. Thus the operators of large scales are always able to establish reputation. The paper sheds light on the existence of professional firms and middlemen.

Key Words: Reputation and Scale   Professional Firm   Middlemen

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1 Introduction

The paper tries to shed lights on two questions. First, in the industry of professional services, like law services and accounting, each task is independent and accomplished by a small group of the professionals. There seems little complementarities between the taskes here, at least,

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less complementaries than the cases in manufacturing industries. For example, a lawsuit is generally similar, but hardly complement, to another of the same category; auditing GM is paralell to auditing GE, but the car bodies of Ford have to fit with the engines. Nevertheless, the professionals still form big firms, such a "Big Four" accounting and auditing firms. What is the economy driving them to work together under one name? Second, the capitalist world is dominated by middlemen, such as supermarkets, department stores and investment banks. They add multiple values, one of which, according to a common wisdom, is that they lend reputational capital to their upper stream clients (hereinafter they will be called "producers" and the down stream clients "buyers"). This lending is generally very costly to the producers. Why do they not establish their own reputation rather than borrow it from the middlemen? Is that because they are not able to do so, or because doing so is more costly?

To answer the questions, the paper presents a theory on the link between scale and reputation. The point is that in market having a large scale facilitates, and sometime is even necessary, to establish reputation. Therefore, since reputation matters a lot in the industries of professional services, the need to establish it glues the professionals together into big firms. And if a producer cannot attain a large enough scale due to technological constraints (for example, the financial need of a firm, or the demand for a specific type of goods, is always limited), those who do not produce and are thus free of the contrainsts are able to reach the necessary scale, through buying goods from many producers and then selling to buyers. That is, they become middlemen.

The economy of the paper consists of N producers and even more buyers, and infinite periods. Each producer yields only one widget each period. Even if he invests in improving its quality, it could still fail to perform with probability $1-q$, as if no investment were made. Thus the failure could be due to bad luck or shirking to investment, and is thus a noisy signal of the investment decision. This noisiness problem alone does not necessarily frustrate the producer establishing reputation, if he was in a context of repeated interaction with a fixed buyer. Reputation can be sustained by the simple mechanism in which the producer promises to refund wholly the buyer in case of failure and the buyer will sever the relationship whenever refunding is defaulted. Unfortunately, in the market that is defined by arm-length tradings this repeated-interaction context are not feasible. In other words, arm-length tradings impose additional constraints upon reputation establishing. One way, which is adopted here, to capture this market constraint is to assume that buyers live for one period only, which could mean that buyers do not have frequent demand for the widgets. Disabled by the double troubles of noisy signals and the market constraint, an individual producer is futile to establish his own reputation if $1-q$, which measures the seriousness of noisiness problem, is big.

The scale makes magic, first by the Law of Large Numbers (LLN). If a middleman is operating in a large scale, by the LLN the average performance of the widgets sold by her (hereinafter the femal pronouns are used to refer a middleman) is very informative on their average quality. The number of performing goods should be around q times the number of high quality widgets. Therefore, the LLN easily removes the trouble of noisy signals, and clears the way to establish reputation. This argument is straightforward, but misses the hardcore problem, the incentive problem of the middleman to take in as many high quality but expensive widgets as supposed in equilibrium. If in an equlibrium she is supposed to sell 1,000 high quality widgets, it seems quite fine if she instead buys in 999 high quality and one low quality one. If this is true for any large number, then in equlibrium there would be no large numbers of high quality widgets and the LLN gets no chances to play its magic. Fortunately, we have Central Limit Theorem (CLT). Applying CLT to analyze the marginal incentive of the middleman taking more high quality widgets, the paper ensures the existence of large numbers of high quality widgets in equilibrium. Therefore, the middleman, saved by both the LLN and CLT, is always able to establish reputation if the scale is large enough. Moreover, the paper finds that the strategy of the market is an important part of the magic. Basically, the market sets a critical line on the number of performing widgets. If number is over the line, the reputation of the middleman is continued. The line shall not be too high, since then she will fail to achieve it even if trying her best; and it shall not be too low, since then she has no incentive to buy in many high quality widgets. The paper shows that given the scale, within some range the higher is the critical line, the more high quality widgets does she buy in and the better is the equilibrium; in the socially best equilibrium, the line is set at the level that all the widgets sold by her are of high quality. Let the scale goes to infinity, so that the critical line of the best equilibrium goes to infinity, the probability of the reputation being continued goes to 1.

The idea that pooling improves informativeness by the LLN is first applied by Diamond (1984) in addressing the viability of banks. Here the idea is applied to discuss the link between reputation and scale. Moreover, that paper does not apply CLT to consider the incentive problem in case of large numbers (though Diamond (1996) considers the case where $N = 2$) or consider the strategy of the market. On the existence of big professional firms, several papers, such as Bowles & Skogh (1989) and Morrison & Wilhelm (2003), also presents the viewpoint that partnership is an institution resolving the asymmetric information problem. But they do not consider the the channel analyzed in this paper through the LLN and CLT. The literature on middlemen is relatively thicker, such as Rubinstein & Wolinsky (1987), Johri & Leach (2002) and Rust & Hall (2003), but all focuses on the effects of middlemen on search and match friction, not from their advantage in establishing reputation. I do not think that capture all of the values added by them. If the only problem is to match buyers with sellers, that could be simply solved by setting up a meeting place, such as markets or exchanges. Moreover, the development of World Wide Web and Electronic platforms can accomodate a large volume of agents to meet together at low costs. However, we do not seemingly see the decline of middlemen. The literature on brand stretching, such as Wernerfelt (1988), Choi (1998), Andersson (2000) and Cabral (2000), considers the benefits of "pooling sale", that is, selling two kinds of widgets under one brand-name. But in those papers, the reputation is given and they do not consider the effects of pooling sale in feeding back to the reputation development. Actually, the mechanisms in those papers does not work for the case of two independent and indentical (iid) widgets, whereas that is the case considered here.

The rest of paper is organized as follows. Section 2 presents the basic model. In the section 3 the paper shows the first point that individual producers are impossible to establish reputation in the market if noisiness problem is serious enough. Section 4 presents the second point that in the market a middleman is able to establish reputation no matter how noisy the signal is, if the scale of her business is large enough. Section 5 concludes and discusses some questions about the competition between middlemen and their alternative organizations.

2 The Basic Model

The economy consists of N producers of widgets and much more buyers, and lasts for infinite periods. The producers live for ever. They interact with buyers in the widget market. Market here is defined by arm-length tradings. To capture it well and to distinguish it clearly from the context of long relationships between fixed agents, I assume buyers live for one period only. That may mean that buyers do not have frequent demand for the widgets; either a widget is a durable good, or of such types that having consumed one today you will not demand another in the near future, like the service for lawsuits and architectural design. That does not seem to capture well the auditing service demanded quarterly by public firms. Nevertheless the constraint imposed by the market here is actually "no refunding constraint", as is clear later, which I think also applies to the auditing services market.

At the start of each period, each producer yields one widget. He decides whether to invest I in improving its quality or not. The investment gives rise to high quality and failing to make it leads to low quality. However, even a high quality widget fails to perform with probability $1-q$; low quality one never performs. When it performs, it gives utility v to the consumer; otherwise, it gives 0. The realization of the performance is independent across widgets. After all widgets are yielded out, they are sold to buyers in the market. Each buyer demands at most one widget. Then the buyers consume it and retire out of the economy. Subsequently, the next period starts.

The buyer of a widget never observes its quality. However, after consuming it he knows its performance, hence the value (v or 0), which is then also observed by the producer. Actually it is observed only by the two and thus is never contractible. Afterwards, however, the buyer spreads honestly the performance information into the market before retiring so that it is known to the agents of the next period. This assumption is grounded on the consideration of his human nature. He and his fellows are humans. Humans like to communicate their experience, personally undergone or heard from others, with their fellows. Morevoer, they prefer being honest when disinterested, though often lie in hope of benefits. In a word, the performance of widgets is not contractible, but observed to the market[1]. Then the performance history of a producer's widgets

[1] This information structure is also assumed in Holmstrom (1999), Tadelis (1999) and Wang (2007).

up to period $t-1$, denoted by h^{t-1}, is observed to the market of period t. Denote by $h_t = s$ (success) or f (failure) the period t performance. Then $h^t = h_1 h_2 ... h_t$.

The performance of a widget is a signal of the investment. If it performs, then $\tilde{I} = 1$. But failure to perform is a nosiy signal. It may be because of bad luck, or $\tilde{I} = 0$. Intuitively, the bigger q is, the less probably the reason is the former. In this sense q measures the informativeness of failure.

All agents are risk neutral. The one period discount is $\beta < 1$.

Assumption 1: $qv > I$.

Thus, it is socially efficient to invest. The question is whether the producers are able to establish reputation of providing high quality goods.

Assumption 2: $I \geq q^2 v$.

The assumption means that q is small, and thus that signal "failure" is noisy enough.

This basic model will be enriched in Section 4, to incorporate an agent of the third type, the middleman. The next section shows that the individual producers cannot establish the reputation of their own under the assumption. That is exactlly why the economy needs the middleman to improve the efficiency.

3 No Reputation with Single Performance

In the market, because there are more buyers than the producers, the market clearing price must be set at such a level that the buyers are indifferent to buy. Let $\widetilde{q}$ be the probability of a widget performing. Then the price of a producer's widget at period t is

$$p_t = E(\tilde{q}|h^{t-1}, p_t)v \qquad \text{(Market Constraint)}$$

where the available information consists of the history of the producer, h^{t-1}, and the price itself.

This equation actually characterizes the market and is thus called the "Market Constraint". Basically it means that all tradings are in one-off arm-length way, and particularly there is no refunding for $h_{t-1} = f$ through $p_t < E(\tilde{q}|h^{t-1}, p_t)v$. The reason in the economy is clear. The buyer of yesterday does not need the widget any more. Neither is the refunding able to be arranged by direct ways; h_{t-1} is not observed by a court (or any third party) and you shall never expect the buyer or the producer to tell the truth when both are interested. Thus the market constraint is actually "No Refunding Constraint". It also holds in the markets where the buyers have repeated demand but the producers have all the market power to select among the equilibria[2]. I think that that is the case for the market for auditing service demanded by public companies.

Market constraint alone does not always thwart the producers' endeavour to establish reputation. Consider the following simple mechanism. Buyers trust the quality of a producer's widget in this period and thus pay qv for it if and only if his widget of the last period performs; otherwise, they will forever pay only 0 afterwards. A producer in the trust mode has incentive to invest if $\frac{q\beta}{1-q\beta}(qv - I) > I$, because besides β, the discounting has to take into account the continuation probability, which is q if he has invested. Let β go to 1, and the inequality becomes $q^2 v > I$. In other words, if q is so big as to invalidate Assumption 2, which means the noisiness problem is not so serious, a producer can enjoy reputation at least for a while, even under the market constraint. Unfortunately, by Assumption 2, that is not the case.

On the other hand, without the market constraint, the noisiness problem alone cannot prevent a producer establishing reputation at all. Suppose there were not the market constraint and a producer were in repeated interaction with a fixed buyer. Then the following mechanism would give the producer the incentive to invest when β is close to 1. $p_0 = v; p_t = v$ if $h_{t-1} = s$, but 0 if $h_{t-1} = f$, or if the producer ever defaults to deliver the widget at price 0 after a failure. A simple argument of Folk Theorem shows that the producer will never default when β is close to

[2]Suppose a equilibrium E requires a producer to refund the buyer in some period. Then the producer can profitablly deviate as follows. He defaults refunding, but finds another buyer to start a new relationship. If the producer has the power to select the equilibrium, then he will choose the equilibrium best to him to play with the new buyer. Thus the continuation value to the producer in this best equilibrium is no smaller than that in equilibrium E; furthermore in this period he gains the should-be-refunded money. Overall, the deviation is profitable. Therefore, any equilibrium does not involve refunding in case of the producers having all the bargaining power, and the market constraint gets in again.

1. Then if a producer invests, next period he gets v with probability q; otherwise he definitely gets 0. That is, he invests if $\beta qv > I \Leftrightarrow qv > I$ if $\beta = 1$, Therefore, even though Assumption 2 presents, a producers is able to establish reputation, if there were not market the constraint.

When crashed by the double burdons of the market constraint and noisiness problem, individual producers in the economy are hopeless to establish repuation, as the proposition below shows.

Proposition 1 *Under both the market constraint and Assumption 2, there is no equilibrium where a producer ever invests in some period.*

The proof is put in the appendix.

Thus, in no period the socially efficient investment is made in the economy, which is very inefficient.

4 The Reputation of a Middleman

The efficiency will be improved if some agent is capable of observing all widgets' quality. Suppose this capable agent becomes a middleman, buying from the producers and selling to buyers. The nice thing of becoming a middleman is being released from the technological constraints upon production, thus able to operate in a large scale. I will show below that if the scale is big enough then she can always establish reputation. Then, the social efficient investment is made. Moreover, the middleman reaps fat benefits from her reputation, which gives strong incentive for the agents of the economy to invest in the capability and ensures the existence of the capable agents.

Here I skip the analysis of human capital investment and suppose some capable agents are already there. For the convenience of exploration, in this section I put aside the problem of competition between the capable agents and study the case with only one capable agent in the economy, supplementing the set-up of Section 2, and denote her by M. In addition, I put aside the problem of how the capable agent is organized and suppose she becomes a middleman, buying from the producers and selling to buyers. Both problems of competition and of organization will be discussed later in the next section.

To slip around Moore-Repullo way of using the information that is known to two parties, it is further assumed that the middleman and a producer could arrange secret side payment between them for collusion,and particularly that the actual buying prices of the middleman are unobservable to buyers.

Therefore, there is one middleman in the economy. She naturally has all the bargaining power over both the producers and buyers. Thus the buying price of low quality widgets is $p_l = 0$ and that of high quality widgets is $p_h = I$. Now the moral hazard problem is shifted upon M. She would like to buy in cheap low quality widgets but sell them out as if they were of high quality. Can she establish the reputation of selling high qualilty widgets? She can, if the scale is large enough. The point is that when she sells many widget at one period, their average performance is very informative about the average quality by the LLN, which removes the noisiness problem of the individual producers and thus enable her to establish reputation. This argument looks straightforward, but it misses the hardcore problem here, the equilibrium incentive of M to sustain as large number of high quality widgets as supposed by the market. Suppose in equilibrium it expects that she sells 1000 high quality widgets. It looks quite fine if she sells instead 999 high quality widgets but hide one low quality among them. If this argument could apply to any large numbers, then in equilibrium there would not be a large number of high quality widgets, and the LLN has no chance to exert its magic effect. Thus the key is to anaylze the marginal benefit of M buying in more high quality widgets; what works here is not the LLN, but Central Limit Theorem (CLT).

Without loss of any generality, I assume all widgets are intermediated by M and thus nothing is sold in the direct-sale market[3]. So M sells N widgets each period. N is supposed to be a large number. The state of the economy in each period is decided by the number of successfully performing widgets, which is public information at the end of the period. Buyers' strategy could

[3]By the argument of Section 3, only low quality widgets are sold in direct-sale market. Whether some low quality widgets, if there are, are intermediated by M does not matter to the discussion here.

be conditional on the historical states in a complex way. This complexity in game theory side does not add much to the core argument here, however. Thus, here they are supposed to use the following trigger strategy. They start from trusting M and accepting price $p = \alpha qv$, where α is the believed share of high quality widgets among the total N ones. If the number of performing widgets in this period is no less than some critical value j, buyers keep trusting M and accept price αqv in the next period; if less than j widgets perform, the buyers will distrust her for ever and accept only $p = 0$ afterwards. As shown below, this strategy could be a part of equilibria. Moreover, for some carefully chosen j, in equilibrium $\alpha = 1$, that is, all the widgets are invested in trust mode, which is first best in some sense.

Given that strategy of buyers, M decides how many of the N widgets are of high quality. Denote by k the number of high quality goods. In equilibrium $\alpha = \frac{k}{N}$ in trust mode. Surely in distrust mode $k = 0$.

Consider M's decision of k in trust mode. Denote by V the value to M from the start of a period in trust mode. First, let the continuation value V be given. For any $k < j$, there can never be no less than j performing widgets. Thus M has zero chance to keep her reputation (namely the trust of the buyers) getting the continuation value. For $k \geq j$, notice that since N is large, the critical value j is large and hence $k \geq j$ has to be large. By CLT, the number of performing widgets, X_k, is approximately distributed as a Norman distribution with mean kq and variance $kq(1-q)$, denoted by $N(kq, kq(1-q))$. Thus, the event that X_k is no less than j happens with probability $\Phi(\frac{kq-j}{\sqrt{kq(1-q)}})$, where Φ is the culmulative distribution function of standard normal variables. If the event happens, M keeps the reputation in the next period and gets the continuation value V; if it fails to happen, M lose the reputation forever and get 0 afterwards. Therefore, M's problem in trust mode is

Problem 1 $\max_k b(k) = \left\{ \begin{smallmatrix} f(k) \equiv \Phi(\frac{kq-j}{\sqrt{kq(1-q)}})\beta V - kI, \; if. N \geq k \geq j \\ -kI, \; if \; j < k \end{smallmatrix} \right\}.$

Denote by $k^*(j;V)$ the solution of the problem, which surely exists and by P^* the probability of the reputation being continued.

On the other hand, given M's decision rule $k^*(j;V)$, the rational expectation of the market dictates $\alpha = \frac{k^*}{N}$. So from each period in trust mode, M gets $N \cdot \alpha qv - k^* I = k^*(qv - I)$. And the next period keeps in trust mode with probability P^* and the future is discounted with rate β. Therefore,

$$V = \frac{k^*(qv - I)}{1 - P^*\beta} \tag{1}$$

An equilibrium is characterized by the profile $\{j; k^*, V\}$ such that given V, $k^* = k^*(j; V)$, and that given k^*, V is decided by (1). It is trivial that for any j, $k^* = V^* = 0$ is an equilibrium. However, we are more interested in Reputation Equilibrium, defined as follows.

Definition 1 *An equilibrium is called "**Reputation Equilibrium**", if* $k^* > 0$.

The purpose of the section is analyze when an Reputation Equlibrium exists.

Obviously, for Problem 1 the solution of maximizing the second branch of $b(k)$ is $k = 0$. Denote by $\widehat{k}(j; V)$ the solution of the problem of maximizing the first branch of $b(k)$, that is, maximizing $f(k)$ subject to $N \geq k \geq j$. $P^* = 0$ if M chooses the second branch and equals $\widehat{P} \equiv \Phi(\frac{\widehat{k}q - j}{\sqrt{\widehat{k}q(1-q)}})$ if M chooses the first branch. M chooses this branch if and only if $f(\widehat{k}) \geq 0$. Then

$$k^* = \begin{Bmatrix} \widehat{k}, \text{ if } \beta\widehat{P}V - \widehat{k}I \geq 0 \\ 0, \text{ if otherwise} \end{Bmatrix} \tag{2}$$

Therefore, a Reputation Equilibrium is a profile $\{j; \widehat{k}, V\}$ such that

(1): Given V, $\widehat{k} = \widehat{k}(j; V)$;

(2): Given $\widehat{k}$, V is decided by (1) for $k^* = \widehat{k}$ and $P^* = \widehat{P} \equiv \Phi(\frac{\widehat{k}q - j}{\sqrt{\widehat{k}q(1-q)}})$;

(3): $\beta\widehat{P}V - \widehat{k}I \geq 0$.

Condition (3) is actually the individual rationality constraint (IR). Conditions (1) and (2) are the incentive compatibility constraint (IC), which captures the hardcore problem of the existence of large numbers. V is decided by the market expectation of the number of high quality widgets. So conditions (1) and (2) exactly say that optimally M picks as many high quality widgets as exactly expected by the market, even this is a large number. I will first address the IC constraint and then go to the IR constraint.

For that purpose, properties of $\widehat{k}(j; V)$ are found first. Let us consider the following problem.

Problem 2 $\max_k f(k) \equiv \Phi(\frac{kq-j}{\sqrt{kq(1-q)}})\beta V - kI$, *if.*$N \geq k \geq j$.

For the convinience of exploration, let us first fix j, which is actually the market's strategy, and suppose N is big enough relative to j such that the constraint $N \geq k$ is not binding in Problem 2. Later I discuss the best j and come back to that constraint.

$f'(k) = \beta V \Phi'(\frac{kq-j}{\sqrt{kq(1-q)}})\frac{qk+j}{2\sqrt{k^3q(1-q)}} - I$[4]. So the first order condition (FOC) of Problem 2 is

$$\Phi'(\frac{kq-j}{\sqrt{kq(1-q)}}) = \frac{I}{\beta V}\frac{2\sqrt{k^3q(1-q)}}{qk+j} \tag{3}$$

For the second derivative, we have

Lemma 2 *If* $j > 6(1-q)$, $f''(k) = 0$ *has unique root* $\widetilde{k}$ *and* $f''(k) > 0$ *for* $k < \widetilde{k}$ *and* $f''(k) < 0$ *for* $k > \widetilde{k}$.

Proof. Put in the appendix. ∎

The lemma means $f'(k)$ has an unique maximizer $\widetilde{k}$, which is independent of V, and is increasing before it and decreasing after it. $\max_k f'(k) = f'(\widetilde{k})$ is an increasing function of V. Let $\widehat{V}$ be the value of V such that $\max_k f'(k) = 0$. Then if $V < \widehat{V}$, $f'(k) < 0$ always. The solution of Problem 2 is $\widehat{k} = j$. The continuation probability is $\widehat{P} = q^j \approx 0$, the probability with which all j widgets perform. Therefore, $\beta\widehat{P}V - jI \approx -jI < 0$. Thus $k^* = 0$. If $V > \widehat{V}$, $f'(k) = 0$ has two roots, $k_1 < \widetilde{k} < k_2$. The bigger one, k_2, is a local maximizer. But the solution of problem 2 may still instead be $\widehat{k} = j$. Then $k^* = 0$. If $k^* > 0$, $\widehat{k} = k_2$ is the solution of Problem 2 and therefore, $k^* = \widehat{k} = k_2$. Let us consider when this is the case.

Lemma 3 *Suppose* $k^* > 0$. *Then* $k^* = k_2 > \frac{j}{q}$ *if* $j > \frac{2\pi(1-q)}{q^2}$ *and* $k_1 < \frac{j}{q}$.

Proof. Put in the appendix. ∎

The first part of the lemma is interesting by itself. By the LLN, we may be inclined to think that if M wants to sustain reputation, then she will only choose $\frac{j}{q}$ high quality goods to make the number of performances exactly overcome the critical value j. But the lemma says that she will buy in more high quality goods to obtain some cushion. Let us do the variable

[4] In the paper, "′" is used to denote the first order derivative and "″" the second order.

transformation $t(k) = \frac{kq-j}{\sqrt{kq(1-q)}}$, which is strictly increasing and thus has the inverse function, denoted by $k(t)$. Let $g(t) = \frac{I}{\beta V}\frac{2\sqrt{k^3q(1-q)}}{qk+j}|_{k(t)}$. Then the FOC becomes $\Phi'(t) = g(t)$. The two functions are illustrated in figure 1 below. From it it is straightforward that $t(k_2) > 0$, so that $k_2 > \frac{j}{q}$, and that $t(k_1) < 0$, so that $k_1 < \frac{j}{q}$.

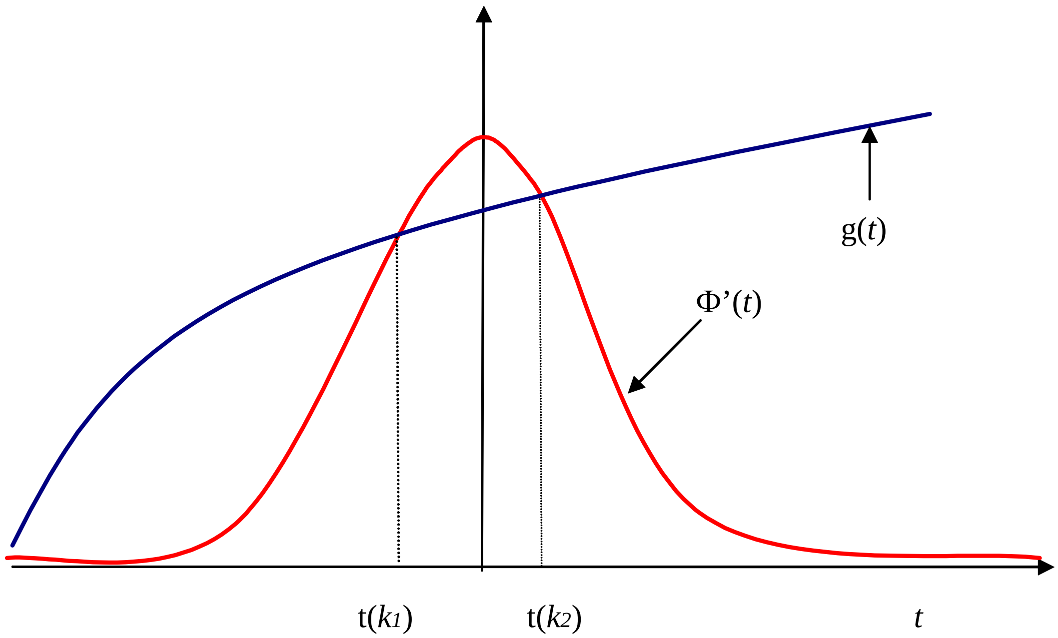

Figure 2: The solutions of (3)

Therefore, if $\{j; \widehat{k}, V\}$ is a Reputation Equilibrium, then $\widehat{k}$ is the solution of (3) that is bigger than $\frac{j}{q}$. Then the IC, that conditions (1) and (2), is equivalent to the existence of the solution (k, V) of (3) and (1) that satisfies $k > \frac{j}{q}$. Such a pair of (k, V) is called "*consistent pair*".

Lemma 4 *A consistent pair* (k, V) *exists if* $j > 2(1-q)\pi[\frac{I(1-0.5\beta)}{\beta(qv-I)}]^2$.

Proof. Put in the appendix. ■

The proof of this lemma really depends on CLT. Substitute (1) into (3), we have $\Phi'(\frac{kq-j}{\sqrt{kq(1-q)}}) = \frac{I(1-\Phi\beta)}{\beta(qv-I)}\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)}$. Do again the variable transformation $t(k) = \frac{kq-j}{\sqrt{kq(1-q)}}$. The right hand side becomes $h(t) =: \frac{I(1-\Phi(t)\beta)}{\beta(qv-I)}\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)}|_{k=k(t)}$, while the left hand side is $\Phi'(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$, the dense function of standard normal distribution. The requirement $k > \frac{j}{q}$ is equivalent to $t > 0$. Thus the existence problem of a consistent pair becomes whether $\Phi'(t) = h(t)$ has a positive solution. The nice thing is that for given j at the neighboorhood of $k = \infty$, $t = \frac{kq-j}{\sqrt{kq(1-q)}} \sim k^{\frac{1}{2}}$ and $\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)} \sim k^{-\frac{1}{2}}$. Thus, when $t \to \infty$, $h(t) \sim t^{-1} \gg \frac{1}{\sqrt{2\pi}}e^{-t^2/2} = \Phi'(t)$. Then we only need $h)0) < \Phi'(0)$ to ensure that $\Phi'(t) = h(t)$ has a positive solution.

Thus the (IC) constraint is satisfied. I guess that given a j, there is a unique consistent pair. Nevertheless in case of multiple solutions, let us focus on the one that has the highest k. Denote it by $k(j)$. Then the continuation value $V(j) = \frac{k(j)(qv-I)}{1-P(j)\beta}$, where $P(j) = \Phi(\frac{kq-j}{\sqrt{kq(1-q)}})|_{k=k(j)}$. And the IR, that is, condition (3), becomes $\beta P(j)V(j) - k(j)I \geq 0$. The constraint $N \geq k$ not binding in Problem 2 is equivalent to the condition that $N \geq k(j)$. Lemma 2 hints that $k(j)$ is an increasing function of j. That is indeed the case as shown below.

Lemma 5 $k'(j) > 0$ *for* $j > \max(\frac{I^2(1-q)}{[\beta(qv-I)\Phi'(1)]^2}, \frac{4(1-q)(qv+I)^2}{[\beta(qv-I)]^2})$.

Proof. Put in the appendix. ■

with j increasing, there are actually double reasons for k to increase with j. One is is that for given V, the standard for the reputation to be continued is increased and thus M has to buy in more high quality widgets to meet the higher standard (higher j). This effect could be seen from (3) (the FOC). On the other hand, higher j brings about higher continuation value V because the market expects more high quality widgets, which is clear from (1).

We turn to the IR and figure out when $\beta P(j)V(j) - k(j)I \geq 0$ for some j. For that purpose, the following properties of $P(j)$ is helpful.

Lemma 6 $P'(j) > 0$ *and* $\lim_{j\to\infty} P(j) = 1$.

Proof. Put in the appendix. ■

That is, when j (thus N) is large, M has incentive to buy in such a large number of high quality widgets that the reputation will almost surely be continued. The intuitive reason is that the relative marginal cost of increasing the continuation probability P is in the order $\frac{1}{\sqrt{j}}$ (thus $\frac{1}{\sqrt{N}}$). For example, by CLT, $\Pr ob(X_k \geq qk) = 0.5$. But $\Pr ob(X_k \geq qk - 3.4\sqrt{kq(1-q)}) = 0.9998$, where X_k is the number of the performing widgets among k high quality widgets. That is, given a j, if M takes $k = \frac{j}{q}$ then the continuation probability is 0.5. But if $k = \frac{j}{q} + \frac{3.5\sqrt{(1-q)j}}{q}$, which guarantees $qk - 3.4\sqrt{kq(1-q)} > j$ for big j, then the continuation probability is bigger than 0.9998. The marginal cost of raising the continuation probability from 0.5 to 0.9998, measured as increment in the number of high quality widgets, is in order of $\sqrt{j}$, but the continuation value V, which measures the marginal benefit of raising the probability, is in order of k (thus j), by (1). That is, the relative marginal cost of raising the continuation probability is in order of $\frac{1}{\sqrt{j}}$. This decreasing relative marginal cost ensures that the continuation probability goes to 1, when j goes to infinity.

This lemma actually solves out the incentive problem of M. When $P(j) \to 1$, then $\frac{V(j)}{k(j)} \to \frac{qv-I}{1-\beta}$, by (1). Here is the place for our main result:

Proposition 2 *For any β close to 1 enough such that $\frac{\beta(qv-I)}{1-\beta} > I$, Reputation Equilibrium exists for big enough N.*

Proof. *: The (IC) constraint is satisfied by lemma 4. The IR is $\beta P(j)\frac{V(j)}{k(j)} - I \geq 0$. As $\lim_{P(j)\to 1} \frac{\beta V(j)}{k(j)} = \beta\frac{qv-I}{1-\beta} > I$, by lemma 6 $\lim_{j\to\infty} \frac{\beta V(j)}{k(j)} = \beta\frac{qv-I}{1-\beta} > I$. Thus there exists some j_0 such that $\beta P(j_0)\frac{V(j_0)}{k(j_0)} > I$ for all $j \geq j_0$. Particularly, $\{j_0, k(j_0), V(j_0)\}$ is a Reputation Equilibrium. Let $N_0 = k(j_0)$. Then this equilibrium exists for all $N > N_0$. Actually for a given $N > N_0$, any $\{j, k(j), V(j)\}$ for a $j \geq j_0$ is a Reputation Equilibrium so long as $N \geq k(j)$.* ■

The proposition says that the middleman can always establish reputation if the scale is big enough, no matter how serious the noisiness problem (how small q) could be. The miracle is made via two channels, through the LLN and CLT respectively. Let us consider the following scenaria. M promises that all N goods are of high quality. How to judge the credibility of the promise based on X_N, the number of performances? By the LLN, if M does what he promises, then almost certainly $\frac{X_N}{N} \approx q$. That is, X_N is a very informative signal of the aggregate quality. On the other hand, it depends on CLT to arrange the incentive scheme based on X_N, namely to design a proper critical value j. If $j = qN$, then the continuation probability is approximately one half even if all N widgets are of high quality, which could be too low to satisfy the participation constraint. Nevertheless, if the buyers relax the standard by setting $j = qN - 3.4\sqrt{Nq(1-q)}$, then $\Pr ob(X_N > j) = 0.9998$ if all N goods are of high quality. This continuation probability is indeed high enough to satisfies the IR, but might be too loose in the sense of giving M incentive to default on a small number of widgets. Thus we need CLT to set an appropriate j. Moreover, CLT teaches us the decreasing relative marginal costs of raising the continuation probability, which ensures it goes to 1, as shown by Lemma 5.

In discussing the existence of Reputation Equilibrium above, I suppose that N is large enough so that constraint $N \geq k$ is never binding in Problem 2. In other words, if M wishes, she can always buy in a necessarily large number of high quality widgets. The proof of Proposition 2 actually shows that for a given $N > k(j_0)$, there are multiple Reputation Equilibria, each characterized by a critical value j. The following proposition characterizes the social best equilibrium. Since buyers always break even, the social welfare is measured by the continuation value $V(j)$. That is, the best equilibrium is characterized by j that maximizes $V(j) = \frac{k(j)(qv-I)}{1-P(j)\beta}$.

Proposition 3 *For a given N, the best j satisfies $k(j) = N$.*

Proof. $V(j) = \frac{k(j)(qv-I)}{1-P(j)\beta}$ is increasing with $k(j)$ and $P(j)$, both increasing with j by lemmas 5 and 6 when $N \geq k(j)$. So $V(j)$ is increasing with j until $k(j) = N$. If we continue to increase j, then $k(j) = N$ because constraint $N \geq k$ begins binding in Problem 2, but $P(j) = \Phi(\frac{Nq-j}{\sqrt{Nq(1-q)}})$ decreases. Thus $V(j)$ actually decreases. So $V(j)$ is maximized at the j that $k(j) = N$. ■

Remark: The argument above is based on large numbers so that applying CLT is appropriate. However, the point is that there is benefit of diversification and to materialize this benefit it depends on an appropriate j, while large numbers makes the benefit of diversification more striking: M is *always* able to establish reputation. Here I illustrate the point with the case where $N = 2$. Suppose first $j = 1$, that is, buyers keep trusting M if one of the two widgets performs. Given this, M decides to take in $k = 0, 1,$ or 2 high quality widgets. Let P_k be the corresponding continuation probability. Then $P_0 = 0; P_1 = q$ and $P_2 = 2q - q^2$. Let us check when both widgets are of high quality in equilibrium and for simplicity put $\beta = 1$. In the equilibrium the continuation value is $V = \frac{2(qv-I)}{1-P_2}$. The IC (here the constraint for M to buy in two rather one high quality widgets) is $P_2V - 2I \geq P_1V - I \Leftrightarrow$

$$\frac{2q}{1-q} \geq \frac{I}{qv-I} \tag{4}$$

And the IR (here the constraint for M to buy in two rather none high quality widgets) is $P_2V - 2I \geq 0 \Leftrightarrow \frac{2q-q^2}{(1-q)^2} \geq \frac{I}{qv-I}$, which is looser than (4) as $2q - q^2 > 2q(1-q)$. Therefore, if (4) holds true, then both widgets are of high quality in trust mode. Notice that (4) is equivalent to $q^2v \geq \frac{1+q}{2}I$, which could be consistent with Assumption 2. That is, if $\frac{I}{q} \geq qv \geq \frac{1+q}{2q}I$, M is still able to establish reputation of only selling high quality widgets. In other words, the little diversification here ($N = 2$) brings about some room for repuation establishing.

The room above depends on the fact that $j = 1$ is the critical line. Suppose the line is $j = 2$. Then the continuation probabilities $P_0 = P_1 = 0; P_2 = q^2$, and the continuation value is $V = \frac{2(qv-I)}{1-P_2}$. Then the IC constraint is $q^2V - 2I \geq P_1V - I$, which is weaker than the IR constraint, $q^2V - 2I \geq 0 \Leftrightarrow \frac{q^2}{1-q^2}(qv - I) \geq I \Leftrightarrow q^3v \geq I$, which is contradictory with Assumption 2. Thus if $j = 2$, the standard is too harsh and M cannot establish repuation interacting with the line.

Summarily, even the minimum diversification increases the room of reputation establishing under the appropriate critical line set by the buyers.

5 Conclusion and Discussions

This paper presents a theory that it requires necessarily big scale to establish reputation in market. The theory can be applied to understand the existence of professional firms and middlemen. The logic is that by operating in a large scale, a middleman transforms the time series data of individual producer's performances into sectional data, which increases the informativeness of the ex post observation by the LLN. This argument is straightforward, but it overlooks the incentive problem of the middleman. Given the scale is large, her shirking in a small number of widgets hardly affect aggregate performance and thus may not be dected by the market. For example, even if the market expects 1000 high quality widgets from the middleman, it seems fine that she only supplies 999 high quality but one low quality widgets. If that is the case, it will break the equilibrium expectation of the market. By applying Central Limit Theorem the paper shows that the expectation of large numbers is sustained. Thus she can always establish reputation if the scale is large enough, and the probability of the reputation being continued goes to 1 when the scale goes to infinity. Moreover, the strategy of the market, the critical line, is an important part of the mechanism. Withine some range, a higher line leads to a better equilibrium. Under the best line, all widgets are of high quality when the reputation is continued.

Here I discuss further some questions.

The Effect of Competition: Why Big Four or Three only?

In the last section I assume that there is only one middleman in the economy. Then she is able to establish reputation if the scale of her business is large enough. As a result, she makes huge profit (actually gaining the whole surplus of the economy). That will seduce other agents to invest in the precious human capital of observing the quality of the widgets, so as to become middlemen also. Thus competition is unavoidable. What is the effect of the competition? For simplicity, let us suppose here the cost of human capital investment, that is the entry cost, is neglectable. Then, how many middlemen can we expect for the economy? Would the free entry drive the profit of these middlemen down to 0?

The question of the effect of competition is very relevant today. Now we are worried that the rating industry or the high end of accounting and auditing industries lacks the necessary competition because it is oligopolied by Big Three or Four[5]. It is surely very hard to argue for a specific number to be the "social best". Rather, the point here is that in these industries we cannot expect much from competition even if the entry is free. The reason is that in these industries, reputation is a key element to do the business; to establish reputation, a large scale is necessary.

Suppose there are L middlemen in the economy and they share the whole business scale N. Consider first the case without price competition. Then by Proposition 2, each middleman needs the minimum scale N_0 to be able to establish reputation. Thus the scale of each middle $\frac{N}{L}$ has to be no smaller than N_0 for him to be able to establish reputation. That is, $L \leq \frac{N}{N_0}$. What

[5]See "BDO Chief eyes challenge to Big Four", *Financial Times*, November 12.

happens if they compete with intake prices p? Then the profit from each high quality widget will be $\pi = qv - p < qv - I$. Notice that the smaller π is, the bigger minimum scale is required to establish reputation. Actually, if p is continuous, then the equilibrium will present *One* middleman only with such a profit level π that the minimum scale is N. So we shall be already happy with Big Four or Three, rather than Big One! This shall not be surprising at all, because there is positive externality among selling each widget in the sense of increased informativeness; the bigger the scale is, the better positioned a middleman is to establish reputation. In an economy with positive externality, natural monopolist will happen.

The Organization of the Capable Agent

In the last section, I assume the clever agent who is capable of observing the quality of the widgets will become a *middleman*, that is, she buy in from the producers and then sell out to the buyers. However, this is an unjustified assumption. Being a middleman is just one way of organizing the human capital of the capable man. Let us call this organization "Middleman". We can imagine the following two other organizations. In one organization, she does not put in her own capital to buy in widgets; instead, she rate each widget to be good or bad. A good rating is sold to the producer at price $qv - I$ and a bad rating at price 0. This organization can be called "Rating Agency". In the other organization, she not only put in her own capital but also hire the producers as her employees. That is, she is running a big professional firm as the senior partner who does not engage in production but specializes in monitoring only. This organization can be called "Partnership". Overall, we have three organizations: Middleman, Rating Agency, and Partnership.

Within this model, these three organizations cannot be distinguished. It would be very interesting if we can find a set-up in which all the three can arise and be distinguished in the equilibrium. To distinguish "Middleman" from "Rating Agency" could be easier as the difference is whether M puts in her own capital. Some friction of financial market can explain the advantage of putting the capital; or if at the start the identity of the capable agents are not common knowledge, then using their own capital to buy in the widgets is really a signal of the capability. I think it is really difficult to distinguish "Middleman" from "Partnership"; it is actually a question of the boundary of the firm and thus always hard to break through.

6 Appendix

The lengthy proofs are put here.

6.1 The Proof of Proposition 1

Let $V(h)$ be the instantaneous value of a producer *after* history h, which is a sequence of "s" (successes; value v is generated) and "f" (failures; the value is 0). Let $p(h)$ be the price of his widget of the current period, which can only depends on h. I am going to prove that if $q^2v \leq I$, then for any h, $V(h) = 0$. As buyers always get 0, that means no social surplus is generated in any period, that is, no investment is made in any period.

If the producer invests in this period, then with probability q his widget performs and thus generates value $V(hs)$ from the begining of the next period; with probability 1-q, it fails and the continuation value is $V(hf)$. Thus $V(h) = p(h) - I + \beta q V(hs) + \beta(1-q)V(hf)$.

If the producer does not invest, his widget definitely fails. Thus $V(h) = p(h) + \beta V(hf)$.

Comparing the two values, the producer invests if and only

(a1): $\beta q(V(hs) - V(hf)) \geq I$.

By market constraint, $p(h) = E(\widetilde{q})v = \{ {}^{qv \text{ if (a1) holds}}_{0 \text{ if (a1) does not hold}}$. Incorporating this into the value funcation, we have the following in equilibrium:

(a2): $V(h) = \{ {}^{qv - I + \beta q V(hs) + \beta(1-q)V(hf) \text{ if (a1) is true}}_{\beta V(hf) \text{ if (a1) is not true}}$.

In an equilibrium where the producer has incentive to invest in at least one period, $V(h) > 0$ for some h. Therefore to prove proposition 1, it suffices to prove that $V(h) = 0$ for any h.

Surely for any equilibrium and any h, $V(h) < \frac{qv}{1-\beta}$. Thus $V^* = \sup\{V(h)|\text{any equilibrium, any } h\}$ exists. The proposition is proved if I can show $V^* = 0$ if $qv - \frac{I}{q} \leq 0$. For the simplicity here, I assume V^* is reached in some equilibrium and some history, that is, $V^* = V(h)$ for some h in some equilibrium. For the general case, I only need to use the fact that $\beta < 1$.

Suppose otherwise $V^* > 0$. Then first the second branch of (a2) does not hold. Otherwise, $V(h) = \beta V(hf) \leq \beta V^* < V^*$, contradicting with $V^* = V(h) > 0$. Thus:

$V(h) = qv - I + \beta qV(hs) + \beta(1-q)V(hf) \leq qv - I + \beta qV(hs) + \beta(1-q)(V(hs) - \frac{I}{\beta q}) = qv - \frac{I}{q} + \beta V(hs) \leq qv - \frac{I}{q} + \beta V(h)$, where the first inequality is from (a1) and the second from the fact that $V(h) = V^* = \sup\{V(h)\} \geq V(hs)$. However, if $qv - \frac{I}{q} \leq 0$, $qv - \frac{I}{q} + \beta V(h) \leq \beta V(h) < V(h)$ (because $V(h) > 0$), contradicting the inequality above.

Therefore I have proved that $V^* = 0$ if $qv - \frac{I}{q} \leq 0$. Q.E.D.

6.2 The Proof of Lemma 2

$f'' = \Phi''[\frac{qk+j}{2\sqrt{k^3q(1-q)}}]^2 + \Phi'\frac{-(qk+3j)}{2\sqrt{k^5q(1-q)}}$. Applying $\Phi''(x) = -x\Phi'(x)$, $f'' = \Phi'\{\frac{j-kq}{\sqrt{kq(1-q)}}[\frac{qk+j}{2\sqrt{k^3q(1-q)}}]^2 - \frac{(qk+3j)}{2\sqrt{k^5q(1-q)}}\} = \Phi'\frac{1}{4\sqrt{k^7q^3(1-q)^3}}[(j-kq)(qk+j)^2 - 2q(1-q)(qk+3j)k]$. Let $S(k) = (j-kq)(qk+j)^2 - 2q(1-q)(qk+3j)k = -q^3k^3 - q^2k^2(j+2-2q) + [qj^2 - 6q(1-q)j]k + j^3$. Then $S' = -3q^3k^2 - 2q^2(j+2-2q)k + qj^2 - 6q(1-q)j$. Since $S(0) > 0$ and $S(+\infty) = -\infty$, $S(k)$ has a positive root. If $qj^2 - 6q(1-q)j > 0$, then $S'(0) > 0$ and $S''(k) < 0$ for $k \geq 0$. Denote by $\widetilde{k}$ the smallest positive root of $S(k)$. Since $S(0) > 0$, we have $S'(\widetilde{k}) \leq 0$. Because $S''(k) < 0$, we have $S'(k) < 0$ for all $k \geq \widetilde{k}$. Thus $S(k) < S(\widetilde{k}) = 0$ for all all $k > \widetilde{k}$. That is, $\widetilde{k}$ is the unique positive root of $S(k)$, and hence of f'' if $qj^2 - 6q(1-q)j > 0 \Leftrightarrow j > 6(1-q)$. Q.E.D.

6.3 The Proof of Lemma 3

If $k^* > 0$, then $k^* = k_2 > j$. k_2 and also k_1 satisfy the first order condition $\Phi'(\frac{kq-j}{\sqrt{kq(1-q)}}) = \frac{I}{\beta V}\frac{2\sqrt{k^3q(1-q)}}{qk+j}$. Let $t(k) = \frac{kq-j}{\sqrt{kq(1-q)}}$ and its inverse function be $k(t)$. Then the lemma is equivalent to $t(k_2) > 0$. Both $\frac{kq-j}{\sqrt{kq(1-q)}}$ and $\frac{2\sqrt{k^3q(1-q)}}{qk+j}$ are a increasing function of k. Then both $k(t)$ and $g(t) =: \frac{I}{\beta V}\frac{2\sqrt{k^3q(1-q)}}{qk+j}|_{k=k()}$ are increasing with t. And $g(+\infty) = +\infty$.

The first order condition becomes $\Phi'(t) = g(t)$. It has two solutions $t(k_1)$ and $t(k_2)$. And $t(k_1) < t(k_2)$. To prove $t(k_2) > 0$ it suffices to show that $g(0) < \Phi'(0) = \frac{1}{\sqrt{2\pi}}$. $k^* > 0$ if and only if $\beta\widehat{P}V \geq k_2 I \Rightarrow V > \frac{k_2 I}{\beta} > \frac{jI}{\beta}$ ($k_2 > j$ because reputation is possibly sustained only if it holds). Because $t(k) = 0 \Leftrightarrow k = \frac{j}{q}$, $g(0) = \frac{I}{\beta V}\frac{2\sqrt{k^3q(1-q)}}{qk+j}|k = \frac{j}{q} = \frac{I}{\beta V}\frac{\sqrt{j(1-q)}}{q}$. Applying $V > \frac{jI}{\beta}$, we have $g(0) < \frac{I}{\beta}\frac{\beta}{jI}\frac{\sqrt{j(1-q)}}{q} = \sqrt{\frac{1-q}{q^2 j}}$, which is less then $\frac{1}{\sqrt{2\pi}}$ if $j > \frac{2(1-q)\pi}{q^2}$.

For the second part, refer back to the proof of lemma 2: $f''(k) = \Phi'\{\frac{j-kq}{\sqrt{kq(1-q)}}[\frac{qk+j}{2\sqrt{k^3q(1-q)}}]^2 - \frac{(qk+3j)}{2\sqrt{k^5q(1-q)}}\}$. So $f''(\frac{j}{q}) < 0$. By lemma 2, $\widetilde{k} < \frac{j}{q}$. Thus $k_1 < \widetilde{k} < \frac{j}{q}$. Q.E.D.

6.4 The Proof of Lemma 4

(2) means equivalently that k satisfies FOC $\Phi'(\frac{kq-j}{\sqrt{kq(1-q)}}) = \frac{I}{\beta V}\frac{2\sqrt{k^3q(1-q)}}{qk+j}$ with the additional requirement that $k > \frac{j}{q}$ by lemma 3. Remember $P^* = \Phi(\frac{k^*q-j}{\sqrt{k^*q(1-q)}})$. So by (1) $V = \frac{k(qv-I)}{1-\Phi\beta}$. Therefore, the existence of a consistent pair is equivalent to the existence of a solution of the following equation that is bigger than $\frac{j}{q}$:

(a41): $\Phi'(\frac{kq-j}{\sqrt{kq(1-q)}}) = \frac{I(1-\Phi\beta)}{\beta(qv-I)}\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)}$.

As in the last subsection, given j, let $t(k) = \frac{kq-j}{\sqrt{kq(1-q)}}$. Then the left hand side of (a41) is $\Phi'(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$. And denote the right hand side of (a41) by $h(t) =: \frac{I(1-\Phi(t)\beta)}{\beta(qv-I)}\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)}$. Then the problem becomes whether and when the equation $\Phi'(t) = h(t)$ has a solution that is positive.

We know $\frac{kq-j}{\sqrt{kq(1-q)}}$ is increasing with k and when k goes to infinity, $t = \frac{kq-j}{\sqrt{kq(1-q)}} \propto \sqrt{k}$. Thus when k goes to infinity, $\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)} \propto k^{-0.5} \propto t^{-1} \gg e^{-t^2/2}$. And $1 - \Phi\beta > 1 - \beta > 0$. Therefore, when $t \to \infty$, $h(t) \gg \Phi'(t)$. Then to gurantee that $\Phi'(t) - h(t)$ has a positive root, it suffices to show that $h(0) < \Phi'(0) = \frac{1}{\sqrt{2\pi}}$. $h(0) = \frac{I(1-\Phi\beta)}{\beta(qv-I)}\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)}|_{k=\frac{j}{q}} = \frac{I(1-0.5\beta)}{\beta(qv-I)}\sqrt{\frac{1-q}{j}}$ (here I use the fact that when $t = 0$, $\Phi(t) = 0.5$. Then $h(0) < \Phi'(0) \Leftrightarrow j > 2(1-q)\pi[\frac{I(1-0.5\beta)}{\beta(qv-I)}]^2$. Q.E.D.

6.5 The Proof of Lemma 5

Refer back to the proof of lemma 4. $k(j)$ is an implicit function defined by equation (a41) satisfying $k(j) > \frac{j}{q}$. Then we use implicit function theorem to decide the sign of $\frac{dk(j)}{dj}$. Let $C = \frac{2\sqrt{q(1-q)}I}{\beta(qv-I)}$ and $t(k,j) = \frac{kq-j}{\sqrt{kq(1-q)}}$. Then (a41) is equivalent to

$F(k,j) = (qk+j)\Phi'(t(k,j)) - C(1-\Phi(t(k,j)\beta)k^{\frac{1}{2}} = 0$. And $k'(j) = -\frac{\partial F/\partial j}{\partial F/\partial k}$.

First, $\frac{\partial F}{\partial j} = \Phi'(t) + (qk+j)\Phi''(t)\frac{\partial t}{\partial j} + C\beta k^{\frac{1}{2}}\Phi'(t)\frac{\partial t}{\partial j} = \Phi'(t) - (qk+j)t\Phi'(t)\frac{\partial t}{\partial j} + Ck^{\frac{1}{2}}\Phi'(t)\frac{\partial t}{\partial j} = \Phi'(t) - \Phi'(t)\frac{\partial t}{\partial j}[(qk+j)t - C\beta k^{\frac{1}{2}}]$, where the second equality applies $\Phi''(t) = -t\Phi'(t)$. $\Phi'(t) > 0$, $\frac{\partial t}{\partial j} < 0$. I am going to show $(qk+j)t - Ck^{\frac{1}{2}} > 0$ under the conditions of the lemma. Thus $\frac{\partial F}{\partial j} > 0$.

t satisfy (a41) $\Phi'(t) = \frac{I(1-\Phi\beta)}{\beta(qv-I)}\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)} < \frac{2I\sqrt{q(1-q)}}{\beta(qv-I)}\frac{k^{0.5}}{qk+j} < \frac{I\sqrt{q(1-q)}}{\beta(qv-I)}\frac{\sqrt{\frac{j}{q}}}{j} = \frac{I}{\beta(qv-I)}\sqrt{\frac{1-q}{j}}$, where the second inequality applies the facts that $k > \frac{j}{q}$ and that $\frac{k^{0.5}}{qk+j}$ is decreasing with k. When $j > \frac{I^2(1-q)}{[\beta(qv-I)\Phi'(1)]^2}$, then $\frac{I}{\beta(qv-I)}\sqrt{\frac{1-q}{j}} < \Phi'(1)$ and hence $\Phi'(t) < \Phi'(1)$. That means $t \geq 1$ since $\Phi'(t)$ is a decreasing function of t for $t \geq 0$. And $qk + j > qk > Ck^{\frac{1}{2}}$, which holds true if $k > \frac{C^2}{q^2} \Leftarrow \frac{j}{q} > \frac{C^2}{q^2} \Leftrightarrow j > \frac{C^2}{q} \Leftrightarrow j > \frac{4(1-q)I^2}{[\beta(qv-I)]^2}$. Therefore, if $j > \max(\frac{I^2(1-q)}{[\beta(qv-I)\Phi'(1)]^2}, \frac{4(1-q)I^2}{[\beta(qv-I)]^2})$, then $(qk + j)t - Ck^{\frac{1}{2}} > 0$.

Second, $\frac{\partial F}{\partial k} = q\Phi'(t)+(qk+j)\Phi''(t)\frac{\partial t}{\partial k}-\frac{C}{2}(1-\Phi(t)\beta)k^{-\frac{1}{2}}+C\beta k^{\frac{1}{2}}\Phi'(t)\frac{\partial t}{\partial k} = -\frac{C}{2}(1-\Phi(t)\beta)k^{-\frac{1}{2}}-\Phi'(t)[t(qk + j)\frac{\partial t}{\partial k} - C\beta k^{\frac{1}{2}}\frac{\partial t}{\partial k} - q]$. I am going to show that $t(qk + j)\frac{\partial t}{\partial k} - C\beta k^{\frac{1}{2}}\frac{\partial t}{\partial k} - q > 0$. Thus $\frac{\partial F}{\partial k} < 0$. As before, $t > 1$. $\frac{\partial t}{\partial k} = \frac{qk+j}{2\sqrt{k^3q(1-q)}}$. So $C\beta k^{\frac{1}{2}}\frac{\partial t}{\partial k} = C\beta k^{\frac{1}{2}}\frac{qk+j}{2\sqrt{k^3q(1-q)}} = C\beta\frac{qk+j}{2k\sqrt{q(1-q)}} < C\frac{2qk}{2k\sqrt{q(1-q)}} = C\sqrt{\frac{q}{1-q}} = \frac{2\sqrt{q(1-q)}I}{\beta(qv-I)}\sqrt{\frac{q}{1-q}} = \frac{2qI}{\beta(qv-I)}$. Then $C\beta k^{\frac{1}{2}}\frac{\partial t}{\partial k} + q < \frac{2qI}{\beta(qv-I)} + q < q\frac{qv+I}{\beta(qv-I)}$. On the other hand $(qk+j)\frac{\partial t}{\partial k} = \frac{(qk+j)^2}{2\sqrt{k^3q(1-q)}} > \frac{q^2k^{0.5}}{2\sqrt{q(1-q)}} > \frac{q}{2}\sqrt{\frac{j}{1-q}}$, where the first inequality applies $qk+j > qk$ and the second applies $k > \frac{j}{q}$. Therefore, if $\frac{q}{2}\sqrt{\frac{j}{1-q}} > q\frac{qv+I}{\beta(qv-I)} \Leftrightarrow j > 4(1-q)\frac{(qv+I)^2}{[\beta(qv-I)]^2}$, then $(qk + j)\frac{\partial t}{\partial k} - C\beta k^{\frac{1}{2}}\frac{\partial t}{\partial k} - q > 0$. Thus $\frac{\partial F}{\partial k} < 0$.

Summarily, $k'(j) = -\frac{\partial F/\partial j}{\partial F/\partial k} > 0$ if $j > \max(\frac{I^2(1-q)}{[\beta(qv-I)\Phi'(1)]^2}, \frac{4(1-q)I^2}{[\beta(qv-I)]^2}, \frac{4(1-q)(qv+I)^2}{[\beta(qv-I)]^2}) = \max(\frac{I^2(1-q)}{[\beta(qv-I)\Phi'(1)]^2}, \frac{4(1-q)(}{[\beta(qv}$

Q.E.D.

6.6 The Proof of Lemma 6

$P(j) = \Phi(\frac{kq-j}{\sqrt{kq(1-q)}})|_{k=k(j)}$. Refer back to the proof of lemma 4. Let $t(j) = \frac{kq-j}{\sqrt{kq(1-q)}}|_{k=k(j)}$. Then $P(j) = \Phi(t(j))$. $t(j) > 0$ by lemma 2 and is decided by (a41) $\Phi'(t(j)) = \frac{I(1-\Phi\beta)}{\beta(qv-I)}\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)}|_{k=k(j)}$. For the first part of the lemma, notice that $\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)}$ is increasing with k and decreasing with j. By lemma 5, $k'(j) > 0$. Therefore, $\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)}|_{k=k(j)}$ is a decreasing function of j. Then $t(j)$ is increasing with j, because $\Phi'(t)$ is decreasing for $t \geq 0$. Then $P(j) = \Phi(t(j))$ is increasing with j.

For the second part, notice that $\frac{I(1-\Phi\beta)}{\beta(qv-I)} < \frac{I}{\beta(qv-I)}$. By lemma 2, $k(j) > \frac{j}{q}$. Therefore $\lim_{j\to\infty} k(j) = \infty$. And $\lim_{k\to\infty}\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)} = 0$. Thus when j goes to infinity, $\frac{I(1-\Phi\beta)}{\beta(qv-I)}\frac{2\sqrt{k^3q(1-q)}}{k(qk+j)}|_{k=k(j)} \to 0$, that is, $\Phi'(t(j)) \to 0$, which means $t(j) \to \infty$. Then $P(j) = \Phi(t(j)) \to 1$. Q.E.D.

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