More Rules of Natural Deduction for Quantifiers

0. Business Matters:

-- 3rd marked HW will be available for collection from Philosophy UG office later this week. Look for a class email alert.
-- Readings for the final two sessions of term are now available. There was a class email which included the details about this; here is the key information:

   A. Aristotle, Metaphysics, Book IV, Part 4. A public domain translation of this text can be found online: [http://classics.mit.edu/Aristotle/metaphysics.4.iv.html](http://classics.mit.edu/Aristotle/metaphysics.4.iv.html). Please note however that the assignment is only to read the fourth part of the text on this webpage.

   B. Nelson Goodman, "The New Riddle of Induction" in Fact, Fiction and Forecast. Two copies of Goodman’s book are on three-hour loan in the reserves section of Sloman Library.

00. Review: At our last session we began to learn the rules of natural deduction that govern the use of quantifiers. In particular we learned rules for introducing and eliminating the universal quantifier. We referred to these operations by various names:

**Instantiation** is the operation of moving from a generalization (a universally quantified generalization) to one of its instances. We called this the rule for eliminating the universal quantifier, or UE (or \( \forall E \)), since one moves from a universally quantified wff to one that is not universally quantified.

**Generalization** is the opposite movement: from a particular wff (an instance) to its corresponding universally quantified generalization. We called this the rule for introducing the universal quantifier, or UI, (\( \forall I \)).

000. Examples of instantiation and generalization in English:

**GENERALIZATION**

Blair is a politician and a liar.
All politicians are liars.

**INSTANTIATION**

All generalizations are false.
This generalization is false.

0000. Remember: Instantiation is always valid. But generalization must meet special conditions. In particular, one can only validly generalize from an instance if the instance was introduced without depending on premises that mention the individuals whose names occur in the instance.

00000. Review Exercises using UE and UI: Construct natural deduction proofs for the following sequents.

   a) \( \forall x [Fx \rightarrow Gx], \forall x [Fx]; \forall x [Gx] \)
   
   hint: your strategy here should be to instantiate the two generalizations in the premises, run modus ponens, and then generalize from there to the conclusion. Be sure to check dependency numbers.

   b) \(~Rab; \sim \forall x [Rxb]\)
1. The Rule of EI: We have now introduced the two rules governing the use of the Universal Quantifier. What about the Existential Quantifier? Here again there will be an introduction rule and an elimination rule. Start with the Introduction rule. Here the basic idea is very simple. If we start from a formula which attributes a certain property or relation to an individual, then we are warranted in making the correlate existentially quantified claim. Hence for instance if we know that John loves Mary then we can infer that someone loves Mary. In QL, we can move from the formula $Ljm$ to the formula $\exists x \[Lxm\]$. Here is an informal statement of the rule:

EI: Given a formula containing a name on any line of proof you may replace one or all occurrences of that name with a variable. Introduce the existential quantifier to the resulting matrix and write the formula on a new line. Annotate the line ‘EI’ together with the line number of the original line. The dependency numbers of the new line are identical with those of the line of the original formula.

2. Example using EI: Hence to take the simplest case just described:

<table>
<thead>
<tr>
<th>Line</th>
<th>Formula</th>
<th>Rule</th>
<th>Line Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Rab</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\exists x [Rxb]$</td>
<td>EI,1</td>
<td>1</td>
</tr>
</tbody>
</table>

Rab $\vdash \exists x [Rxb]$ 

3. Exercises using EI: Construct Natural Deduction Proofs for the following sequents:

- $\forall x [Fx]: \exists x [Fx]$
- $\forall x [Rax]: \exists x [Rax v Rxa]$
- $\forall y [Gy \rightarrow Hy]: Ga \rightarrow \exists y [Hy]$
- Lab, $\exists x [Lxb] \rightarrow \exists x [Lbx]$: $\exists x [Lbx]$

4. Introducing EE: The Rule for Eliminating the existential quantifier is the final but also the most complex rule of natural deduction. One generally must use it to prove sequents that rely on an existentially quantified premise. The EE rule is an assumption discharge strategy; that is, it involves making an assumption, extracting a conclusion from that assumption, and then discharging the assumption by applying the EE rule.

5. The Strategy of EE: The key to the strategy of EE is that one takes as one’s assumption a ‘typical disjunct.’ (TD). This is roughly analogous to the rule for UI, which worked by generalizing from a perfect exemplar of its kind. Here is the general idea: An Existentially Quantified formula tells me that something (i.e., at least one thing) in the domain of discourse has a certain property or feature. But it doesn’t tell me which of the various entities in the domain has that property. One can thus think of an existentially quantified formula as a long disjunction. For example, $\exists x [Fx]$ effectively means $(Fa v Fb v Fc v Fd \ldots)$. I know that something on that long list of disjuncts has the property of being F, but I don’t know which one. The strategy of EE is to assume of some one of those named individuals that it is the one that has the property in question – it is a “typical disjunct.” But one does not simply move from $\exists x [Fx]$ to Fa; rather one uses Fa as an assumption that is ultimately discharged.
6. An Example:

$$\forall x [Fx \to Gx], \exists x [Fx] : \exists x [Gx]$$

<p>| | | | |</p>
<table>
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<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$$\forall x [Fx \to Gx]$$</td>
<td>PI</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>$$\exists x [Fx]$$</td>
<td>PI</td>
<td>{2}</td>
</tr>
<tr>
<td>3</td>
<td>Fa</td>
<td>Assumption TD</td>
<td>{3}</td>
</tr>
<tr>
<td>4</td>
<td>Fa $$\to$$ Ga</td>
<td>UE, 1</td>
<td>{1}</td>
</tr>
<tr>
<td>5</td>
<td>Ga</td>
<td>MP, 3, 4</td>
<td>(1,3)</td>
</tr>
<tr>
<td>6</td>
<td>$$\exists x [Gx]$$</td>
<td>EI, 5</td>
<td>(1,3)</td>
</tr>
<tr>
<td>7</td>
<td>$$\exists x [Gx]$$</td>
<td>EE, 2, 3, 6</td>
<td>{1, 2}, discharge 3</td>
</tr>
</tbody>
</table>

Notice what happens here. Line (2) tells us that *something* in the domain of discourse has the property of being F. At line (3) we make an assumption – namely that the object named ‘a’ is object that has this property, where a is some arbitrarily chosen member of the domain of discourse, a “Typical Disjunct”. This assumption is *not* warranted by the premises (they don’t tell us which object has that property, only that at least one object does). But we make the assumption for the sake of argument. In this case we then use modus ponens to show that if a is F then it is also G. This entails that something is G. At line 6 we reach this conclusion on the basis of the assumption we have made. Line 7 discharges the assumption.

7. **Note the double occurrence!**: Notice here that the same wff occurs both in line 6 and in line 7. This pattern is common in EE proofs. Why do we need line 7 at all, if we already have the same formula in line 6? Formally, the answer is that line 6 still includes an assumption (3) among its dependencies. A conclusion that still depends on an assumption has not been shown to follow from its premises. This is a general feature in all assumption-discharge strategies. Check the dependency number to make sure that your conclusion depends only on premises! More informally the answer is this: Line 2 introduced the premise saying that *something* in the domain of discourse has the property of being F. At line 3 I suppose that it is the thing named ‘a’ that has that property. The wff at line 6 is showing that, *under this assumption*, the conclusion follows. The final line of the proof is where EE comes in. I am effectively here claiming that conclusion follows from line 2, *no matter which* of the individuals in the domain of discourse is the one that has the property F.

8. **A first exercise using EE**: Yikes; that sounds complicated! But it is not as bad as it sounds. Try an example for yourself. Construct a Natural Deduction Proof for the following sequent:

$$\exists x [Gx], \exists y [Gy]$$

9. **The Restrictions**: Like the Rule for UI, EE has restrictions, which are intended to enforce the requirement that the our assumption is of a genuinely typical disjunct. There are two restrictions:

   a) If the name in the typical disjunct is contained in the conclusion derived for EE then EE cannot be legitimately applied.

   b) If the name in the typical disjunct is itself contained in any premise or assumption used to derive the conclusion for EE from the typical disjunct then EE cannot be legitimately applied to that disjunct.

Essentially both of these restrictions are designed to ensure that one’s reasoning about the individual in one’s assumption is confined to features of that individual that it possesses as a typical member of its class. See Tomassi, 298-300 for some examples of proofs which violate these restrictions.
10. **The Rule of EE:** The full rule for EE is given on page 301 of the textbook. One key feature to note is the annotation of line numbers. When deploying the EE rule, indicate three numbers:
   - the line number of the original existential formula
   - the line number of the typical disjunct
   - the line number of the conclusion derived from the typical disjunct
The dependency numbers of the new line consist of all the dependencies belonging to the derived conclusion, with the following exceptions:
   - omit the line where the typical disjunct was assumed; record this assumption as discharged.
   - include the dependency numbers that appeared on the line where the original existential formula occurred.

11. **Exercises using EE:** Construct Natural Deduction Proofs for the following sequents:

   \[
   \exists x [Fx \land Gx] \land \exists x [Fx] \land \exists x [Gx]
   \]

   \[
   \forall y [Gy \rightarrow Hy] \land \exists x [Gx] \rightarrow \exists y [Hy]
   \]

   \[
   \exists x [-Fx] \land \forall x [Fx]
   \]

**Homework for the Next Session**

**Reading:** Tomassi, P. *Logic*, Review Chapter VI

**Exercises:** Tomassi, Revision Exercises, pp. 328-329